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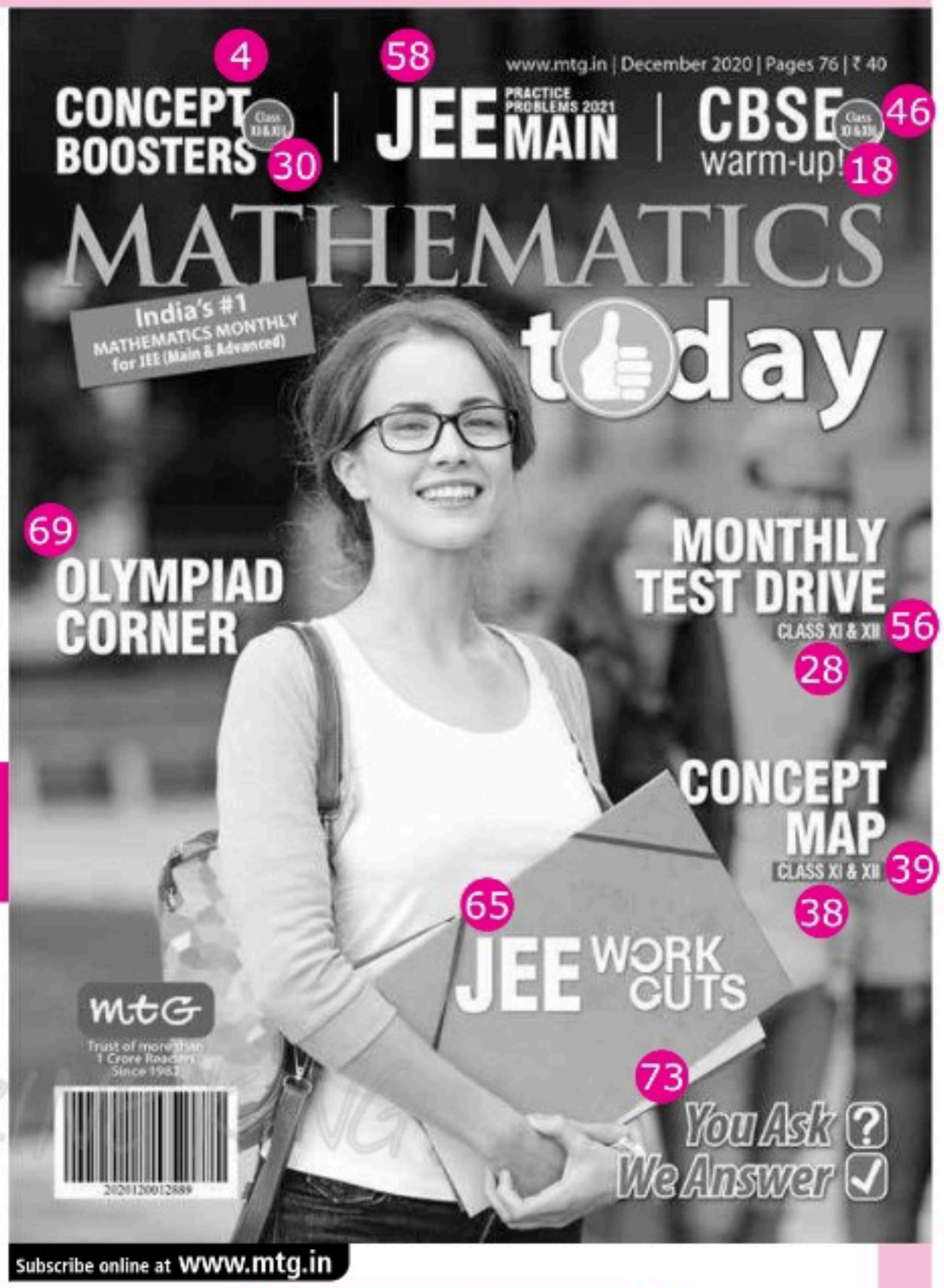

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CONCEPT BOOSTER

Class XI

This column is aimed for students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR

Straight Lines

GENERAL EQUATION OF A STRAIGHT LINE

Any equation of first degree of the form $ax + by + c = 0$, where a, b, c are constants always represents a straight line (at least one out of a and b is non-zero).

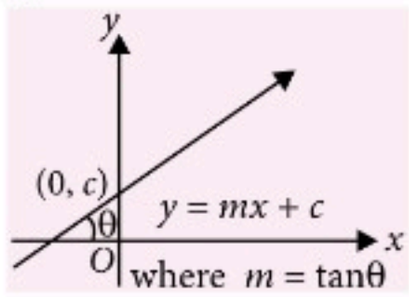
- **Slope or gradient of a line:** If θ is the angle at which a straight line is inclined to the positive direction of x-axis, then $\tan\theta$ is called the slope or gradient of the line, where $0 \le \theta \le 180°$ ($\theta \ne 90°$).
- If a line is parallel to x-axis, then its slope is $\tan 0° = 0$.
- Slope of a line perpendicular to x-axis is not defined.
- Slope of the line passing through two points (x_1, y_1) and (x_2, y_2) is given by $\dfrac{y_2 - y_1}{x_2 - x_1}$.

INTERCEPTS OF A STRAIGHT LINE ON THE AXES

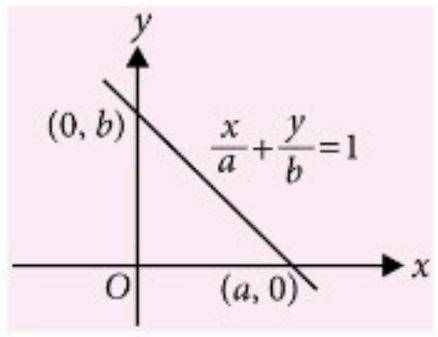
- **Intercept of a line on x-axis :** If a line cut the x-axis at $(a, 0)$, then 'a' is called the intercept of a line on x-axis or the x-intercept. If a line intersect the x-axis at $(-2, 0)$, then the length of intercept is $|-2| = 2$.
- **Intercept of a line on y-axis:** If a line cuts y-axis at $(0, b)$, then b is called the intercept of a line on y-axis or y-intercept.

EQUATION OF A LINE IN DIFFERENT FORMS

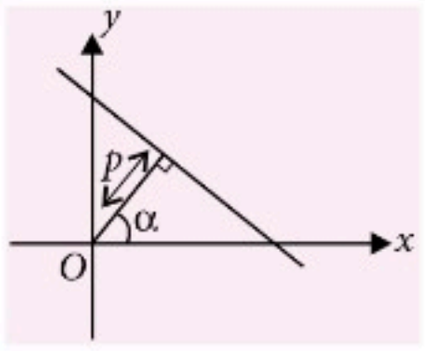
- **Point-slope form :** The equation of a line passing through the point (x_1, y_1) and having slope m is given by $(y - y_1) = m(x - x_1)$.
- **Two point form:** The equation of a straight line passing through two given points (x_1, y_1) and (x_2, y_2) is given by $(y - y_1) = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
- **Slope intercept form:** Equation of a line whose slope is $m(= \tan\theta)$ and which cut an intercept c on the y-axis (*i.e.*, which passes through the point $(0, c)$) is given by $y = mx + c$.



- **Intercept form:** The equation of the line which cuts off intercepts a and b on x-axis and y-axis respectively is given by $\dfrac{x}{a} + \dfrac{y}{b} = 1$.



- **Normal form :** The equation of a straight line upon which the length of perpendicular from the origin is p and the perpendicular makes an angle α with the positive



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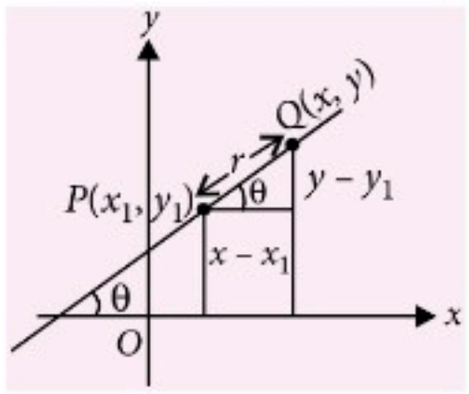
direction of x-axis is given by $x\cos\alpha + y\sin\alpha = p$. In normal form of equation of a straight line, p is always taken as positive and α is measured from positive direction of x-axis in anticlockwise direction between 0 and 2π.

- **Parametric form :** The equation of line passing through a given point $P(x_1, y_1)$ and making an angle θ with the positive direction of x-axis is given by



$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r,$$

where r is the distance between the variable point $Q(x, y)$ and the fixed point $P(x_1, y_1)$.

- Any point on the line will be of the form $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. For different values of r, we will get different points on the line.
- If $P(x_1, y_1)$ is any point on the line which makes an angle θ with the positive direction of x axis, then there will be two points on the line at a distance r from $P(x_1, y_1)$. One will be relatively upward, if r is taken positive and other will be relatively downward, if r is taken negative.

POSITION OF A POINT WITH RESPECT TO A GIVEN LINE

Let the given line be $ax + by + c = 0$, and we have to check whether the point (α, β) lies above the line or below the line. To check first draw a perpendicular line from $P(\alpha, \beta)$ on the x-axis which intersect the given line at Q. x-coordinate of Q will be α and corresponding to α, y-coordinate of Q will be $\beta' = -\left(\frac{a\alpha + c}{b}\right)$.

Now, if $\beta > \beta'$, then the point will lie above the line, if $\beta = \beta'$, then the point will lie on the line and if $\beta < \beta'$, then the point will lie below the line.

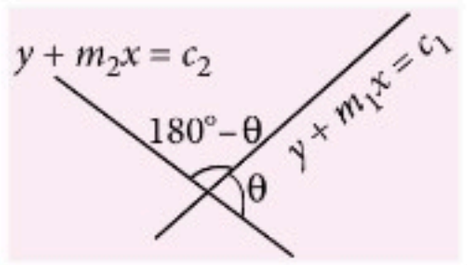
POSITION OF TWO POINTS WITH RESPECT TO A LINE

Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two given points.

Now if $\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} < 0$, and $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of opposite sign, then the points P and Q will lie on the opposite side of the line $ax + by + c = 0$ and, if $\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0$, and $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign, then the points P and Q will lie on the same side of the given line.

ANGLE BETWEEN TWO STRAIGHT LINES

- If θ is the acute angle between two lines, then



$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|,$$

where m_1 and m_2 are the slopes of the two lines and are finite.

- If the two lines are perpendicular to each other, then $m_1 m_2 = -1$.
- Any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$.
- If the two lines are parallel or are coincident, then $m_1 = m_2$.
- Any line parallel to $ax + by + c = 0$ is of the form $ax + by + k = 0$.
- If any of the two lines is perpendicular to x-axis, then the slope of that line is not defined (infinite).
- Let $m_1 = \infty$, then

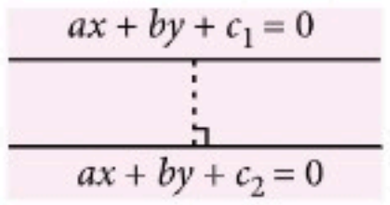
$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{1 - \frac{m_2}{m_1}}{\frac{1}{m_1} + m_2}\right| = \left|\frac{1}{m_2}\right|$$

or $\theta = |90° - \alpha|$, where $\tan\alpha = m_2$ *i.e.*, angle θ is the complimentary to the angle which the oblique line makes with the x-axis.

DISTANCE BETWEEN TWO PARALLEL LINES

The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is

$$\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$



PERPENDICULAR DISTANCE OF A POINT FROM A LINE

The length of the perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

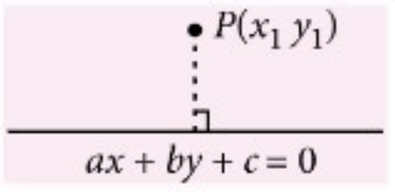
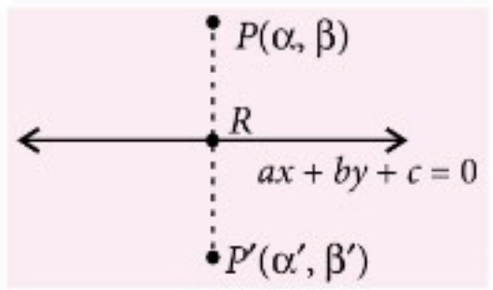


IMAGE OF A POINT W.R.T. A LINE

Let the given line be $ax + by + c = 0$ and we have to find the image of the point $P(\alpha, \beta)$.

Slope of PP' (where P' is image of P) will be b/a, because PP' is perpendicular to given line. Hence equation of PP' is $y - \beta = b/a(x - \alpha)$



Let the point of intersection of line PP' with the given line be $R(p, q)$. Clearly, R is the mid point of PP'.

$$\therefore \frac{\alpha+\alpha'}{2} = p \Rightarrow \alpha' = 2p - \alpha$$

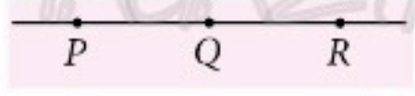
$$\text{Also, } \frac{\beta+\beta'}{2} = q \Rightarrow \beta' = 2q - \beta$$

Hence the image is $(2p - \alpha, 2q - \beta)$.

COLLINEARITY OF THREE POINTS

- Three points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ are collinear, if area of triangle formed by these three points should be equal to zero

$$i.e, \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$



- Any one point of the given three points should divide other two points either internally or externally.

$$\Rightarrow x_3 = \frac{mx_2 + nx_1}{m+n} \text{ and } y_3 = \frac{my_2 + ny_1}{m+n} \text{ for some real } m \text{ and } n.$$

- Slope of PQ = Slope of $QR \Rightarrow \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$

CONCURRENCY OF THREE STRAIGHT LINES

The condition for 3 lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ to be concurrent is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

BISECTORS OF ANGLE BETWEEN TWO GIVEN LINES

Let $a_1x + b_1y + c_1 = 0$ ...(i)
and $a_2x + b_2y + c_2 = 0$ ...(ii) are two intersecting lines. Let $P(x, y)$ be any point on either of the two bisectors of angles of (i) and (ii). Then P is equidistance from (i) and (ii).

$$\Rightarrow \left|\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}\right| = \left|\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}\right|$$

$$\Rightarrow \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad ...(iii)$$

which are the required equations of the two bisectors of angles between (i) and (ii).

If the two given lines are not perpendicular *i.e.*, $a_1a_2 + b_1b_2 \neq 0$, then one of the equations given by (iii) is the equation of the bisector of acute angle and the other that of the obtuse angle.

- Equation of acute and obtuse angle bisectors:
Take one of the given lines and let its slope be m_1 and take one of the bisectors and let its slope be m_2. If θ be the acute angle between them, find $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$.

(i) If $\tan\theta > 1$, then the bisector taken is the bisector of the obtuse angle and the other one will be the bisector of the acute angle.

(ii) If $\tan\theta < 1$, then the bisector taken is the bisector of the acute angle and the other one will be the bisector of the obtuse angle.

- If the constant term c_1 and c_2 in the two equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are of the same sign.

Case-(i) : If $a_1a_2 + b_1b_2 > 0$, then

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

will give the equation of obtuse angle bisector and

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

will give the equation of acute angle bisector.

Case (ii): If $a_1a_2 + b_1b_2 < 0$, then

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

will give the equation of acute angle bisector and $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$ will give the equation of obtuse angle bisector.

MAGAZINE KING

Note : Whether both the lines are perpendicular or not but the angle bisectors of these lines will always be mutually perpendicular.

- The equation of the bisector of the angle which contain a given point:
  The equation of the bisector of the angle between the two lines containing the point (α, β) is $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$
  If $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ are of the same signs or $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ if $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ are of the opposite signs.
- The equation of the bisector of the angle containing the origin:
  Write the equations of the two lines so that the constants c_1 and c_2 are positive. Then the equation $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ is the equation of the bisector containing the origin.
  If $a_1a_2 + b_1b_2 < 0$, then the origin will lie in the acute angle and if $a_1a_2 + b_1b_2 > 0$, then origin will lie in the obtuse angle.

FAMILY OF LINES

Suppose $L_1 = a_1x + b_1y + c_1 = 0$ and $L_2 = a_2x + b_2y + c_2 = 0$ are two intersecting lines and let the point of their intersection be (α, β). Now if we write these two equations in this form $L_1 + \lambda L_2 = 0$ (where λ is a parameter)

i.e., $(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$ ...(i)

then for different values of λ, (i) will give different straight lines.

Hence (i) represent a family of straight lines passing through the point of intersection of $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

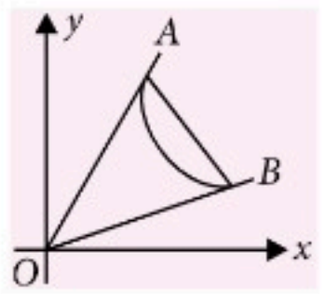
- Family of lines perpendicular to a given line $ax + by + c = 0$ is given by $bx - ay + k = 0$, where k is a parameter.
- Family of lines parallel to a given line $ax + by + c = 0$ is given by $ax + by + k = 0$, where k is a parameter.

PAIR OF STRAIGHT LINES

- The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, represents a pair of straight lines if, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ and $h^2 \geq ab$.

$\Rightarrow \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and $h^2 \geq ab$.

- A homogeneous equation of second degree in x and y, *i.e.*, $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin if $h^2 \geq ab$.



- If the lines through the origin whose joint equation is $ax^2 + 2hxy + by^2 = 0$, are $y = m_1x$ and $y = m_2x$, then $y^2 - (m_1 + m_2)xy + m_1m_2x^2 = 0$ and $y^2 + \frac{2h}{b}xy + \frac{a}{b}x^2 = 0$ are identical, so that $m_1 + m_2 = -\frac{2h}{b}$, $m_1m_2 = \frac{a}{b}$.
- If θ be the angle between two lines, through the origin, then $\tan\theta = \pm\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2} = \pm\frac{2\sqrt{h^2-ab}}{a+b}$
  The lines are perpendicular if $a + b = 0$ and coincident if $h^2 = ab$.
- Joint Equation of Pair of Lines Joining the Origin and the Points of Intersection of a Curve and a Line:
  If the line $lx + my + n = 0$, $((n \neq 0)$ *i.e.*, the line does not pass through origin) cut the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at two points A and B, then the joint equation of straight lines passing through A and B and the origin is given by homogenizing the equation of the curve by the equation of the line
  i.e., $ax^2 + 2hxy + by^2 + (2gx + 2fy)\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$
  is the equation of the lines OA and OB.

ROTATION OF CO-ORDINATE AXES

Let OX, OY be the original axes and OX' and OY' be the new axes obtained after rotating OX and OY through an angle θ in the anticlockwise direction. Let P be any point in the plane having coordinates (x, y) with respect to axes OX and OY and (x', y') with respect to axes OX' and OY'.

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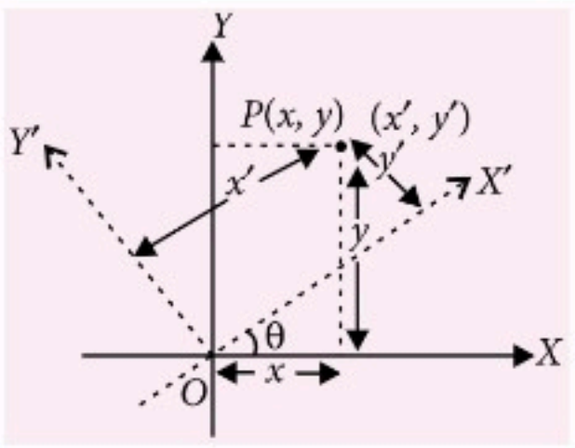


Then $x = x' \cos\theta - y' \sin\theta$, $y = x' \sin\theta + y' \cos\theta$ ...(i)
and $x' = x \cos\theta + y \sin\theta$, $y' = -x \sin\theta + y \cos\theta$. ...(ii)
The above transformation can also be displaced by a table.

	x'	y'
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

If $f(x, y) = 0$ is the equation of a curve then it's transformed equation is $f(x'\cos\theta - y'\sin\theta, x'\sin\theta + y'\cos\theta) = 0$.

PROBLEMS

SINGLE CORRECT ANSWER TYPE

1. Incentre of the triangle formed by the lines $y = x$, $y = 3x$ and $y = 8 - 3x$ is

(a) $\left(\frac{4}{5}, \frac{2-\sqrt{5}}{3}\right)$ (b) $\left(\frac{4}{5}, \frac{2+\sqrt{5}}{3}\right)$

(c) $\left(\frac{4}{3}, \frac{2+2\sqrt{5}}{3}\right)$ (d) $\left(\frac{4}{3}, \frac{2-2\sqrt{5}}{3}\right)$

2. Consider the triangle OAB, where $O = (0, 0)$, $B = (3, 4)$. If the orthocentre of a triangle is $H(1, 4)$, then coordinate of A is

(a) (0, 15/4) (b) (0, 17/4)
(c) (0, 21/4) (d) (0, 19/4)

3. The diagonals of the parallelogram whose sides are $lx + my + n = 0$, $lx + my + n' = 0$, $mx + ly + n = 0$, $mx + ly + n' = 0$ include an angle

(a) $\pi/2$ (b) $\pi/3$

(c) $\tan^{-1}\left(\frac{l^2 - m^2}{l^2 + m^2}\right)$ (d) $\tan^{-1}\left(\frac{2lm}{l^2 + m^2}\right)$

4. Equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines, combined equation of lines that can be obtained by taking the mirror of lines about the x-axis is

(a) $ax^2 + 2hxy + by^2 = 0$
(b) $bx^2 + 2hxy + ay^2 = 0$
(c) $bx^2 - 2hxy + ay^2 = 0$
(d) none of these

5. A variable line drawn through the point (1, 3) meets the x-axis at A and y-axis at B. If the rectangle $OAPB$ is completed, where 'O' is the origin, then locus of 'P' is

(a) $\frac{1}{y} + \frac{3}{x} = 1$ (b) $x + 3y = 1$

(c) $\frac{1}{x} + \frac{3}{y} = 1$ (d) $3x + y = 1$

6. The least value of $|\alpha|$, for which the lines $x = \alpha + m$, $y = -2$ and $y = mx$ are concurrent, is

(a) $\sqrt{2}$ (b) $2\sqrt{2}$ (c) 0 (d) 1

7. The straight lines $L_1 \equiv 4x - 3y + 2 = 0$, $L_2 \equiv 3x + 4y - 4 = 0$ and $L_3 \equiv x - 7y + 6 = 0$

(a) form a right angled triangle
(b) form a right angled isosceles triangle
(c) are concurrent
(d) none of these

8. $P(x, y)$ is called a good point if $x, y \in N$. Total number of good points lying inside the quadrilateral formed by the lines $2x + y = 2$, $x = 0$, $y = 0$ and $x + y = 5$, is equal to

(a) 4 (b) 2 (c) 6 (d) 1

9. The sides of a rectangle are $x = 0$, $y = 0$, $x = 4$ and $y = 3$. The equation of straight line having slope $\frac{1}{2}$ that divides the rectangle into two equal halves is

(a) $2y = x + 1$ (b) $2x = y + 1$
(c) $2y + x = 1$ (d) $2x + y = 1$

10. The equation of the bisector of angle between the lines $x + y = 1$ and $7x - y = 3$ that contains the point (2, 3) is

(a) $3x + y - 1 = 0$ (b) $x - 3y + 1 = 0$
(c) $2x + y - 1 = 0$ (d) $x - 2y + 1 = 0$

11. The locus of point of intersection of lines $x\cos\alpha + y\sin\alpha = a$ and $x\sin\alpha - y\cos\alpha = b$ is (α is a variable)

(a) $2(x^2 + y^2) = a^2 + b^2$
(b) $x^2 - y^2 = a^2 - b^2$
(c) $x^2 + y^2 = a^2 + b^2$
(d) None of these

12. If $x^2 - 7xy - y^2 = 0$, $x + y = 2$, represents the sides of a triangle, then distance between orthocenter and the side of triangle is

(a) 2 (b) $\sqrt{3}$ (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

MAGAZINE KING

13. The equation of an acute angle bisector between lines $(\sqrt{3}-1)y=(\sqrt{3}+1)x$ and $(\sqrt{3}+1)y=(\sqrt{3}-1)x$ is

(a) $y = x$ (b) $x + y = 0$

(c) $y = x + 1$ (d) $y = -x + 1$

14. If the line $y=\sqrt{3}x$ cuts the curve $x^3+y^2+3x^2+8=0$ at the points A, B, C, then $OA\cdot OB\cdot OC$ (O being origin) equals

(a) -32 (b) -64 (c) 108 (d) -100

15. If one vertex of an equilateral triangle of side 2 is the origin and another vertex lies on the line $x=\sqrt{3}y$, then third vertex can be

(a) $(0, 2)$ (b) $(-\sqrt{3},-1)$

(c) $(2, 0)$ (d) $(\sqrt{3},1)$

16. A line passing through the point (2, 2) and the axes enclose an area λ. The intercepts on the axes made by the line are given by the two roots of

(a) $x^2 - 2|\lambda| x + |\lambda| = 0$ (b) $x^2 + |\lambda| x + 2|\lambda| = 0$

(c) $x^2 - |\lambda| x + 2|\lambda| = 0$ (d) none of these

17. Area of the triangle formed by the line $x + y = 3$ and angle bisector of the pair of straight lines $x^2 - y^2 + 2y = 1$ is

(a) 2 sq.units (b) 4 sq.units

(c) 6 sq.units (d) 8 sq.units

18. Let $P\equiv(-1,0)$, $Q\equiv(0,0)$ and $R\equiv(3,3\sqrt{3})$ be three points. Then the equation of the bisector of the angle PQR is

(a) $\frac{\sqrt{3}}{2}x+y=0$ (b) $x+\sqrt{3}y=0$

(c) $\sqrt{3}x+y=0$ (d) $x+\frac{\sqrt{3}}{2}y=0$

19. The incentre of the triangle with vertices $(1,\sqrt{3})$, (0, 0) and (2, 0) is

(a) $\left(1,\frac{\sqrt{3}}{2}\right)$ (b) $\left(2,\frac{1}{\sqrt{3}}\right)$

(c) $\left(\frac{2}{3},\frac{1}{\sqrt{3}}\right)$ (d) $\left(1,\frac{1}{\sqrt{3}}\right)$

20. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in G.P., with the same common ratio, then the points (x_1, y_1), (x_2, y_2) and (x_3, y_3)

(a) lie on a straight line

(b) lie on an ellipse

(c) lie on a circle

(d) are vertices of a triangle

21. Let PQR be a right angled isosceles triangle, right angled at $P(2, 1)$. If the equation of the line QR is $2x + y = 3$, then the equation representing the pair of lines PQ and PR is

(a) $3x^2 - 3y^2 + 8xy + 20x + 10y + 25 = 0$

(b) $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$

(c) $3x^2 - 3y^2 + 8xy + 10x + 15y + 20 = 0$

(d) $3x^2 - 3y^2 - 8xy + 10x - 15y - 20 = 0$

22. If $P \equiv (1, 0)$, $Q \equiv (-1, 0)$ and $R \equiv (2, 0)$ are three given points, then locus of the point S satisfying the relation $SQ^2 + SR^2 = 2SP^2$ is

(a) a straight line parallel to x-axis

(b) a circle passing through the origin.

(c) a circle with the centre at the origin

(d) a straight line parallel to y-axis.

MULTIPLE CORRECT ANSWER TYPE

23. Under rotation of axes through θ, $x\cos\alpha + y\sin\alpha = P$ changes to $X\cos\beta + Y\sin\beta = P$, then

(a) $\cos\beta = \cos(\alpha - \theta)$ (b) $\cos\alpha = \cos(\beta - \theta)$

(c) $\sin\beta = \sin(\alpha - \theta)$ (d) $\sin\alpha = \sin(\beta - \theta)$

24. The angle through which the co-ordinate axes be rotated so that xy-term in the equation $5x^2+4\sqrt{3}xy+9y^2=0$ may be missing is

(a) $\frac{\pi}{6}$ (b) $\frac{-\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

25. Sides of a rhombus are parallel to the lines $x + y - 1 = 0$ and $7x - y - 5 = 0$. It is given that diagonals of the rhombus intersect at (1, 3) and one vertex 'A' of the rhombus lies on the line $y = 2x$. Then the coordinates of the vertex A are

(a) $\left(\frac{8}{5},\frac{16}{5}\right)$ (b) $\left(\frac{7}{15},\frac{14}{15}\right)$

(c) $\left(\frac{6}{5},\frac{12}{5}\right)$ (d) $\left(\frac{4}{15},\frac{8}{15}\right)$

26. If the points $(-2, 1)$, $(3, 4)$ and (a^2, a) lie on the same side of the straight line $6x - 7y - 3 = 0$, then a can lie in

(a) $(0, 1)$ (b) $\left(\frac{6}{5},\frac{7}{5}\right)$

(c) $(2, 4)$ (d) $(-1, 1)$

27. If $x^2 + 2hxy + y^2 = 0$ $(h \neq 1)$ represents the equations of the straight lines through the origin which make an angle α with the straight line $y + x = 0$, then (m_1 and m_2 are slopes of given pair of straight lines)

MAGAZINE KING

(a) $\sec 2\alpha = h$ (b) $\cos\alpha = \sqrt{\frac{1+h}{2h}}$

(c) $m_1 + m_2 = -2\sec 2\alpha$ (d) $\cot\alpha = \sqrt{\frac{h+1}{h-1}}$

ASSERTION & REASON TYPE

Directions : *In the following questions, Statement-1 is followed by Statement-2. Mark the correct choice as :*

(a) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is true, Statement–2 is true, Statement-2 is not a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

28. **Statement-1 :** A chord $y = mx + c$ of the curve $3x^2 - y^2 - 2x + 4y = 0$, which passes through the point $(1, -2)$, subtend a right angle at the origin.

Statement-2 : Lines represented by the equation $(3c + 2m)x^2 - 2(1 + 2m)xy + (4 - c)y^2 = 0$ are perpendicular, if $c + m + 2 = 0$.

29. **Statement-1 :** If algebraic sum of perpendicular distances from $(-2, 0)$, $(3, 1)$ and $(4, 2)$ to the line $ax + by + c = 0$ is zero, then line must pass through $\left(\frac{5}{3}, 1\right)$.

Statement-2 : If algebraic sum of perpendicular distances from $A_i(x_i, y_i)$, $i = 1, 2, 3, ...n$ to $ax + by + c = 0$ is zero, then line must pass through centroid of polygon having vertices at (x_i, y_i).

30. **Statement-1 :** The vertices of a triangle are $A(x_1, x_1\tan\theta_1)$, $B(x_2, x_2\tan\theta_2)$ and $C(x_3, x_3\tan\theta_3)$. If the circumcentre of the triangle ABC coincides with origin and orthocentre $H(x', y')$, then

$$\frac{y'}{x'} = \frac{\sin\theta_1 + \sin\theta_2 + \sin\theta_3}{\cos\theta_1 + \cos\theta_2 + \cos\theta_3}.$$

Statement-2 : In a triangle circumcentre, centroid and orthocentre are collinear.

COMPREHENSION TYPE

Paragraph for Q. No. 31 to 33

In a ΔPQR, with $PQ = r$, $QR = p$, $PR = q$, the cosine values of the angles are given by $\cos P = \frac{q^2 + r^2 - p^2}{2qr}$; $\cos Q = \frac{p^2 + r^2 - q^2}{2pr}$; $\cos R = \frac{p^2 + q^2 - r^2}{2pq}$, and the area of ΔPQR is $\Delta = \frac{1}{2}pq\sin R = \frac{1}{2}qr\sin P = \frac{1}{2}pr\sin Q$.

Let $ABCD$ be a parallelogram whose diagonal equations are $AC \equiv x + 2y - 3 = 0$; $BD \equiv 2x + y - 3 = 0$. If $AC = 4$ units, and area of $ABCD = 8$ sq. units, and $\angle BPC$ is acute, where P is point of intersection of diagonals AC and BD. Then

31. The length of other diagonal BD is

(a) $\frac{10}{3}$ (b) 2 (c) $\frac{20}{3}$ (d) $\frac{11\sqrt{2}}{3}$

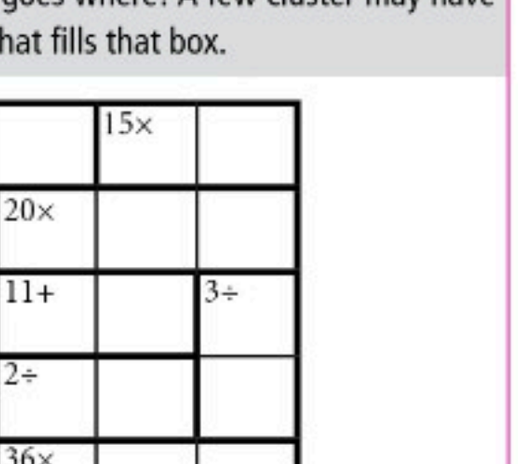
32. The length of side AB is equal to

(a) $\frac{2\sqrt{58}}{3}$ (b) $\frac{4\sqrt{58}}{3}$ (c) $\frac{\sqrt{58}}{3}$ (d) $\sqrt{58}$

33. The length of BC is equal to

(a) $\frac{4\sqrt{10}}{3}$ (b) $\frac{2\sqrt{10}}{3}$ (c) $\frac{8\sqrt{10}}{3}$ (d) $\frac{\sqrt{10}}{3}$

PUZZLE CORNER



MATHDOKU

Introducing MATHDOKU, a mixture of ken-ken, sudoku and Mathematics. In this puzzle 6 × 6 grid is given, your objective is to fill the digits 1-6 so that each appear exactly once in each row and each column.

Notice that most boxes are part of a cluster. In the upper-left corner of each multibox cluster is a value that is combined using a specified operation on its numbers. For example, if that value is 3 for a two-box cluster and operation is multiply, you know that only 1 and 3 can go in there. But it is your job to determine which number goes where! A few cluster may have just one box and that is the number that fills that box.

4–	24×			15×	
	12×		20×		
1–			11+		3÷
1–		120×	2÷		
3÷			36×		
		11+		2–	

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

MATRIX-MATCH TYPE

34. Match the following:

Column-I		Column-II	
A.	The distance between the lines $(x+7y)^2+4\sqrt{2}(x+7y)-42=0$ is	p.	2
B.	If the sum of the distance of a point from two perpendicular lines in a plane is 1, then its locus is $\|x\|+\|y\|=k$, where k is equal to	q.	7
C.	If $6x+6y+m=0$ is acute angle bisector of line $x+2y+4=0$ and $4x+2y-1=0$, then m is equal to	r.	3
D.	Area of the triangle formed by the lines $y^2-9xy+18x^2=0$ and $y=6$ is	s.	1

35. Match the following :

Column-I		Column-II	
A.	The value of k for which $4x^2+8xy+ky^2=9$ is the equation of a pair of straight lines is	p.	3
B.	If the sum of the slopes of the lines given by $x^2-2cxy-7y^2=0$ is four times their product, then the value of c is	q.	–3
C.	If the gradient of one of the lines $x^2+hxy+2y^2=0$ is twice that of the other, then h =	r.	2
		s.	4

SOLUTIONS

1. (c): Equation of one of the bisectors are

$\frac{y-3x}{\sqrt{10}}=\pm\frac{y-x}{\sqrt{2}}$. We have to consider the bisector which is bisecting the internal angle of the triangle. For the internal bisector, the slope of the bisector should be positive.

$$\Rightarrow \left(\frac{1}{\sqrt{5}}+1\right)y-x\left(1+\frac{3}{\sqrt{5}}\right)=0$$

Equation of second bisector is given by

$$\frac{y+3x-8}{\sqrt{10}}=\frac{y-3x}{\sqrt{10}}\Rightarrow x=\frac{4}{3}$$

$$\therefore \text{ Incentre } =\left(\frac{4}{3},\left(\frac{3+\sqrt{5}}{1+\sqrt{5}}\right)\frac{4}{3}\right)=\left(\frac{4}{3},\frac{2+2\sqrt{5}}{3}\right)$$

2. (d): We know that orthocentre is concurrency of altitudes and line joining orthocentre to vertex is perpendicular to the opposite side.

Let $A=(h,k), m_{AH}=\frac{k-4}{h-1}$, $m_{OB}=\frac{4}{3}$,

where m_{AH} and m_{OB} are slopes of lines AH and OB respectively, which are perpendicular to each other.

$$\Rightarrow \quad \frac{k-4}{h-1}\cdot\frac{4}{3}=-1 \Rightarrow 4k+3h=19 \qquad ...(i)$$

Slope of OA *i.e.*, $m_{OA}=\frac{k}{h}, m_{BH}=\frac{4-4}{1-3}=0$

Since OA and BH are mutually perpendicular $\Rightarrow h=0$.

Putting $h=0$ in (i), we get $k=\frac{19}{4}$

Thus coordinate of A is $\left(0,\frac{19}{4}\right)$.

3. (a) : Since the distance between the parallel lines $lx+my+n=0$ and $lx+my+n'=0$ is same as the distance between the parallel lines $mx+ly+n=0$ and $mx+ly+n'=0$. Therefore, the parallelogram is rhombus. Since the diagonals of a rhombus are at right angles, therefore the required angle is $\pi/2$.

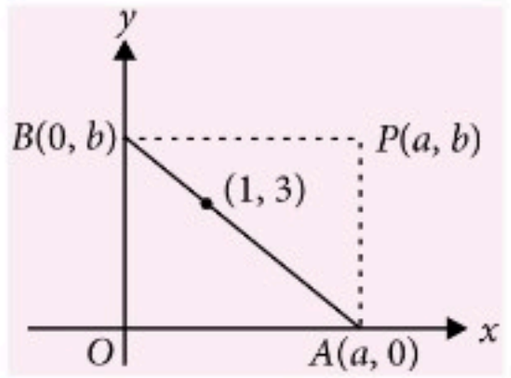
4. (d) : Let the lines represented by $ax^2+by^2+2hxy=0$ be $y=m_1x$ and $y=m_2x$, then $m_1+m_2=\frac{-2h}{b}$, $m_1m_2=a/b$. If these lines are reflected about the x-axis, then their equation becomes $y+m_1x=0$ and $y+m_2x=0$ and their combined equation is

$(y+m_1x)(y+m_2x)=0$

$\Rightarrow \quad y^2+xy(m_1+m_2)+m_1m_2x^2=0$

$\Rightarrow \quad by^2-2hxy+ax^2=0$

5. (c) : Let the line be $\frac{x}{a}+\frac{y}{b}=1$. Since the line is passing through (1, 3),



hence $\frac{1}{a}+\frac{3}{b}=1$.

Now, $A=(a,0)$,

$B=(0,b) \therefore P=(a,b)$

Thus locus of 'P' is

$\frac{1}{x}+\frac{3}{y}=1$

6. (b) : Since the given lines are concurrent. So, point of intersection of lines $x=\alpha+m$ and $y=-2$ which is $(\alpha+m,-2)$ will also satisfy $y=mx$

$\Rightarrow \quad m^2+m\alpha+2=0$

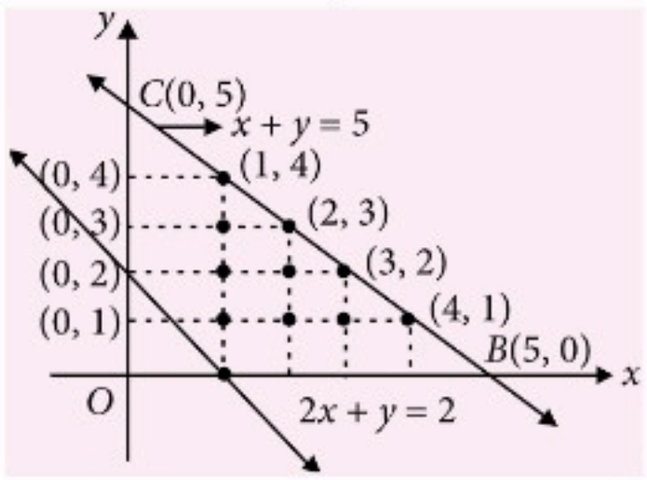
Since m is real $\therefore \alpha^2 - 8 \geq 0 \Rightarrow |\alpha| \geq 2\sqrt{2}$.

Hence the least value of $|\alpha| = 2\sqrt{2}$.

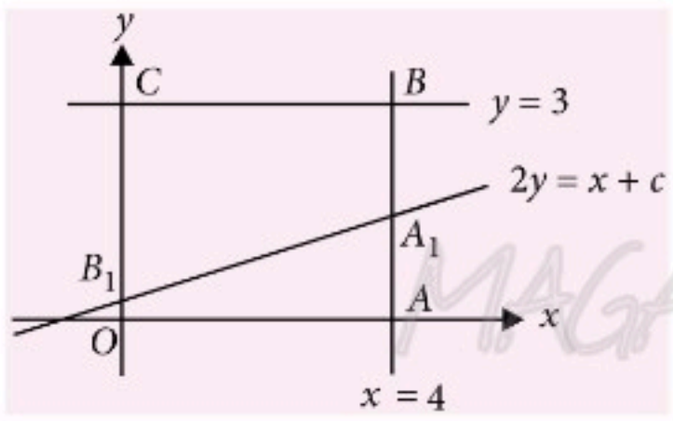
7. (b) : Since the line L_1 and L_2 are perpendicular and angle between L_1 and L_3 is 45°.

Hence, the triangle will be right angled isosceles triangle.

8. (c) : Adjacent figure indicates that there are exactly six good points inside the quadrilateral $ABCD$.



9. (a) : Let the required line be $2y = x + c$.



We have $A_1\left(4, \frac{4+c}{2}\right), B_1\left(0, \frac{c}{2}\right)$

Area of rectangle $OABC$ is 12 sq. units

Thus area of trapezium OAA_1B_1 is

$$6 = \frac{1}{2}(4)\left(\frac{c}{2} + \frac{4+c}{2}\right) \Rightarrow c = 1$$

Thus required line is $2y = x + 1$

10. (b) : Equation of bisector is given by

$$\frac{x+y-1}{\sqrt{2}} = \pm\frac{7x-y-3}{\sqrt{50}}$$

Now at (2, 3), $x + y - 1 > 0$ and $7x - y - 3 > 0$.

Hence, the equation of bisector that contains the point (2, 3) is $\frac{x+y-1}{\sqrt{2}} = \frac{7x-y-3}{\sqrt{50}}$

11. (c) : Let the point of intersection be (h, k)

Then $h\cos\alpha + k\sin\alpha = a$ ...(i)

and $h\sin\alpha - k\cos\alpha = b$ ...(ii)

Eliminating α and β from (i) and (ii), we get required locus as $x^2 + y^2 = a^2 + b^2$

12. (c) : Clearly $x^2 - 7xy - y^2 = 0$ will represent two perpendicular lines *i.e.*, triangle will be a right angled triangle and orthocentre will be the right angled vertex which is (0, 0).

Clearly distance of (0, 0) from $x + y - 2 = 0$ is $\sqrt{2}$.

13. (a) : Given lines are $y = \frac{\sqrt{3}+1}{\sqrt{3}-1}x$ and $y = \frac{\sqrt{3}-1}{\sqrt{3}+1}x$

Clearly inclination of both the lines with the positive direction of x-axis are 75° and 15°. Clearly the inclination of bisector will be 45°. Hence its equation is $y = x$.

14. (b) : Parametric equation of line $y = \sqrt{3}x$ is

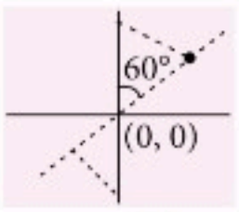
$\frac{x}{\frac{1}{2}} = \frac{y}{\frac{\sqrt{3}}{2}} = r$; Any point on this line is $\left(\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)$

At point of intersection, the point will also satisfy $x^3 + y^2 + 3x^2 + 8 = 0$

$$\Rightarrow \frac{r^3}{8} + \frac{3r^2}{4} + \frac{3r^2}{4} + 8 = 0 \Rightarrow r^3 + 12r^2 + 64 = 0$$

$\Rightarrow OA \cdot OB \cdot OC = -64$

15. (a) : Clearly, the third vertex will lie on the y-axis. Hence the points are (0, 2) or (0, −2)



16. (c) : Let a and b are x and y intercept of line respectively. Then, area = $\frac{1}{2}ab = |\lambda|$ ...(i)

Equation of line passing through (2, 2)

$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{2}{a} + \frac{2}{b} = 1$ ...(ii)

$\Rightarrow 2(a + b) = ab = 2|\lambda|$

$\Rightarrow a + b = |\lambda|$ ...(iii) and $ab = 2|\lambda|$ ...(iv)

$\Rightarrow a(|\lambda| - a) = 2|\lambda|$ [Using (iii) and (iv)]

$\Rightarrow a^2 - a|\lambda| + 2|\lambda| = 0$

Intercepts on axes made by the line are given by $x^2 - |\lambda|x + 2|\lambda| = 0$

17. (a) : The equation $x^2 - y^2 + 2y = 1$ represents $x - y + 1 = 0$ and $x + y - 1 = 0$

Hence, equation of their angle bisectors are $x = 0$ and $y = 1$.

18. (c) : Inclination of QR is 60°. Clearly, angle bisector will be inclined at an angle of 120° with the x-axis.

19. (d) : Since the given triangle is equilateral. Hence incentre will coincides with centroid which is $\left(1, \frac{1}{\sqrt{3}}\right)$.

20. (a) : Since x_1, x_2, x_3 and y_1, y_2, y_3 are in G.P.

$$\Rightarrow \frac{x_2}{x_1} = \frac{x_3}{x_2} = k_1 \text{ and } \frac{y_2}{y_1} = \frac{y_3}{y_2} = k_1$$

MAGAZINE KING

$$\text{Area} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Hence the points are collinear.

21. (b) : The line PQ and PR will be inclined at an angle of 45° with the line $2x + y = 3$

$$\Rightarrow \quad 1 = \left|\frac{m+2}{1-2m}\right| \quad \Rightarrow \quad m = -\frac{1}{3},\ 3$$

Hence the lines are $y - 1 = -\frac{1}{3}(x-2)$ and $y - 1 = 3(x-2)$

Now, we get the equation of pair of lines by multiplying these two equations.

22. (d) : Let the point S be (h, k). According to given condition

$\{(h+1)^2 + k^2\} + \{(h-2)^2 + k^2\} = 2\{(h-1)^2 + k^2\}$

$\Rightarrow \quad 2h + 3 = 0$, which is a straight line parallel to y-axis.

23. (a, c) : Given, $x\cos\alpha + y\sin\alpha = P$

Since, axis is rotated through angle 'θ', then transformed equation is

$\cos\alpha(X\cos\theta - Y\sin\theta) + \sin\alpha(X\sin\theta + Y\cos\theta) = P$

$\Rightarrow X\cos(\alpha - \theta) + Y\sin(\alpha - \theta) = P \Rightarrow X\cos\beta + Y\sin\beta = P$

Hence, $\cos\beta = \cos(\alpha - \theta)$, $\sin\beta = \sin(\alpha - \theta)$

24. (b, d) : $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right) = \frac{1}{2}\tan^{-1}\left(\frac{4\sqrt{3}}{-4}\right)$

$$= \frac{1}{2}\tan^{-1}(-\sqrt{3})$$

$$\Rightarrow \quad \theta = \frac{1}{2}\left(\pi - \frac{\pi}{3}\right), \frac{1}{2}\left(-\frac{\pi}{3}\right) \Rightarrow \theta = \frac{\pi}{3}, -\frac{\pi}{6}$$

25. (a, c) : It is clear that diagonals of the rhombus will be parallel to the bisectors of the given lines and will pass through (1, 3). Equations of bisectors of the given lines are

$$\frac{x+y-1}{\sqrt{2}} = \pm\left(\frac{7x-y-5}{5\sqrt{2}}\right) \text{ or } 2x - 6y = 0,\ 6x + 2y = 5$$

Therefore, the equations of diagonals are $x - 3y + 8 = 0$ and $3x + y - 6 = 0$. Thus the required vertex will be the point where these lines meet the line $y = 2x$. Solving these lines we get possible coordinates as $\left(\frac{8}{5}, \frac{16}{5}\right)$ and $\left(\frac{6}{5}, \frac{12}{5}\right)$.

26. (a, b) : $6a^2 - 7a - 3 < 0 \Rightarrow a \in \left(-\frac{1}{3}, \frac{3}{2}\right)$

27. (a, b, c, d) : Let $x^2 + 2hxy + y^2 = 0$ be given by $y = m_1x$ and $y = m_2x$; $m_1 + m_2 = -2h$

Slope of $y + x = 0$ is -1

$$\tan\alpha = \left|\frac{m_1+1}{1-m_1}\right|, \quad \tan\alpha = \left|\frac{m_2+1}{1-m_2}\right|$$

$$\tan\alpha = \frac{m_1+1}{1-m_1} \text{ and } \tan\alpha = -\left(\frac{m_2+1}{1-m_2}\right)$$

(for +ve signs, in both gives the same value but $m_1 \neq m_2$)

$$\Rightarrow m_1 = \frac{\tan\alpha - 1}{\tan\alpha + 1},\ m_2 = \frac{\tan\alpha + 1}{\tan\alpha - 1}$$

$\Rightarrow \quad m_1 + m_2 = -2\sec 2\alpha \quad \Rightarrow \quad h = \sec 2\alpha$

$$\Rightarrow \quad \cos 2\alpha = \frac{1}{h}, \quad \Rightarrow 2\cos^2\alpha - 1 = \frac{1}{h}$$

$$\Rightarrow \cos\alpha = \left(\frac{1+h}{2h}\right)^{1/2} \Rightarrow \cot\alpha = \sqrt{\frac{h+1}{h-1}}$$

28. (a) : Statement-2 is true as the sum of the coefficients of x^2 and y^2 must be zero for lines to be perpendicular *i.e.*, $3c + 2m + 4 - c = 0 \Rightarrow c + m + 2 = 0$.

In statement-1, let the equation of the chord be $y = mx + c$, then equation of the pair of lines joining the origin to the points of intersection of the chord and the curve is

$$3x^2 - y^2 - 2x\left(\frac{y-mx}{c}\right) + 4y\left(\frac{y-mx}{c}\right) = 0$$

$\Rightarrow \quad (3c + 2m)x^2 - 2(1 + 2m)xy + (4 - c)y^2 = 0$

which are at right angles if $c + m + 2 = 0$ (using statement-2) and since the line $y = mx + c$ passes through $(1, -2)$, $c + m + 2 = 0$. So statement-1 is also true.

29. (a) : $\sum_{i=1}^{3} \frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}} = 0$

$$\Rightarrow a\left(\sum x_i\right) + b\left(\sum y_i\right) + 3c = 0$$

$$\Rightarrow a\frac{\sum_{i=1}^{3} x_i}{3} + b\frac{\sum_{i=1}^{3} y_i}{3} + c = 0 \Rightarrow \frac{5}{3}a + b + c = 0.$$

30. (a) : Since circumcentre is origin and $OA = OB = OC = r$

$$OA = \sqrt{x_1^2 + x_1^2\tan^2\theta_1} = x_1\sec\theta_1$$

$\therefore \quad x_1 = r\cos\theta_1$

Hence, $A(r\cos\theta_1, r\sin\theta_1)$, $B(r\cos\theta_2, r\sin\theta_2)$, $C(r\cos\theta_3, r\sin\theta_3)$.

Also, cicumcentre (O), centroid (G), and orthocentre (H) are collinear.

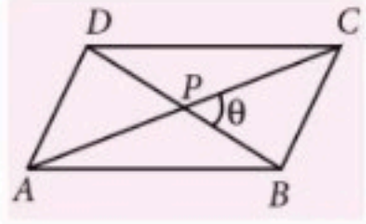
$\Rightarrow$ Slope of OH = Slope of GO

$$\Rightarrow \quad \frac{y'-0}{x'-0}=\frac{(y \text{ coordinate of } G)-0}{(x \text{ coordinate of } G)-0}$$

$$\Rightarrow \quad \frac{y'}{x'}=\frac{\sin\theta_1+\sin\theta_2+\sin\theta_3}{\cos\theta_1+\cos\theta_2+\cos\theta_3}$$

31. (c) **32. (a)** **33. (b)**

Let P be the point of intersection.



$$\tan\theta=\left|\frac{\frac{-1}{2}+2}{1+1}\right|=\frac{3}{4}\Rightarrow \sin\theta=3/5$$

$$\text{Area of } \Delta CPB=\frac{1}{2}PC\cdot PB\sin\theta=2\Rightarrow PB=\frac{10}{3}$$

$$\Rightarrow \quad BD=\frac{20}{3}$$

$$\cos(\pi-\theta)=\frac{-4}{5}=\frac{4+\frac{100}{9}-AB^2}{2\times 2\times\frac{10}{3}}\Rightarrow AB=\frac{2\sqrt{58}}{3}$$

Again from ΔCPB, $BC=\frac{2\sqrt{10}}{3}$

34. A → p; B → s; C → q; D → r

(A) $(x+7y)^2+4\sqrt{2}(x+7y)-42=0$

$$\Rightarrow (x+7y)[x+7y+7\sqrt{2}]-3\sqrt{2}(x+7y)-42=0$$

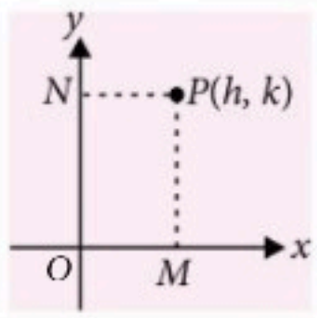
$$\Rightarrow (x+7y)[x+7y+7\sqrt{2}]-3\sqrt{2}(x+7y+7\sqrt{2})=0$$

$$\Rightarrow (x+7y+7\sqrt{2})(x+7y-3\sqrt{2})=0$$

$$\Rightarrow \quad x+7y+7\sqrt{2}=0 \text{ and } x+7y-3\sqrt{2}=0$$

$$\Rightarrow \quad d=\left|\frac{7\sqrt{2}+3\sqrt{2}}{\sqrt{1+49}}\right|=\frac{10\sqrt{2}}{\sqrt{50}}=2$$

(B) Let two perpendicular lines are coordinate axes.



Then, $PM + PN = 1 \Rightarrow \quad h + k = 1$

Hence, the locus is $x + y = 1$.

But if the point lies in other quadrants, then $|x| + |y| = 1$. Hence, value of k is 1.

(C) Angle bisector between the lines $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$ is given by

$$\frac{x+2y+4}{\sqrt{1+4}}=\pm\frac{(-4x-2y+1)}{\sqrt{16+4}}$$

$$\Rightarrow \quad x+2y+4=\pm\frac{(-4x-2y+1)}{2}$$

$$\Rightarrow \quad 2(x + 2y + 4) = \pm (-4x - 2y + 1)$$

Since $aa' + bb' < 0$, so +ve sign gives acute angle bisector. Hence, $2x + 4y + 8 = -4x - 2y + 1$

$$\Rightarrow \quad 6x + 6y + 7 = 0 \Rightarrow \quad m = 7$$

(D) We have, $y^2 - 9xy + 18x^2 = 0$

or $\quad y^2 - 6xy - 3xy + 18x^2 = 0$

$$\Rightarrow \quad y(y - 6x) - 3x(y - 6x) = 0$$

$$\Rightarrow \quad y - 3x = 0 \text{ and } y - 6x = 0$$

The third line is $y = 6$. Therefore, area of the triangle formed by these lines,

$$=\frac{1}{2}\begin{vmatrix}0 & 0 & 1\\ 1 & 6 & 1\\ 2 & 6 & 1\end{vmatrix}=\frac{1}{2}|6-12|=3 \text{ sq. units}$$

35. A → s; B → r; C → p, q

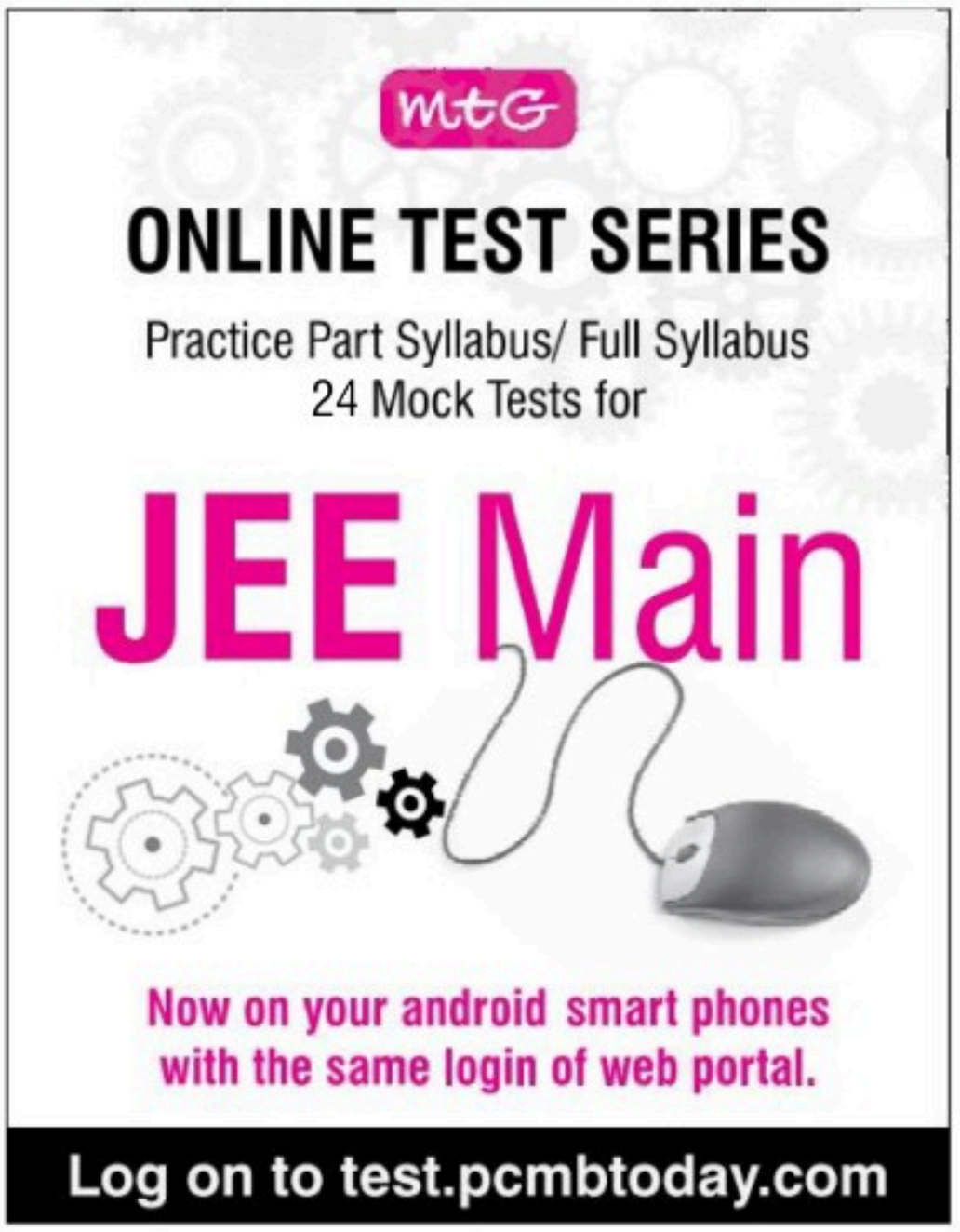
(A) The equation represents a pair of straight lines if $(4)(k)(-9) - (-9)(4)^2 = 0 \Rightarrow \quad k = 4$

(B) Given, $m_1 + m_2 = 4m_1m_2$

$$\Rightarrow \quad \frac{-2h}{b}=\frac{4a}{b}\Rightarrow -2(-c)=4\times 1\Rightarrow c=2$$

(C) We have, $2m_2+m_2=\frac{-h}{2}$, $2m_2^2=\frac{1}{2}$

$$\Rightarrow 2\left(\frac{-h}{6}\right)^2=\frac{1}{2}\Rightarrow h^2=9\Rightarrow h=\pm 3$$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CBSE warm-up!

CLASS-XI

Chapterwise Practice questions for CBSE Exams as per the reduced syllabus and marking scheme issued by CBSE for the academic session 2020-21.

Series 7 **Introduction to Three Dimensional Geometry and Limits and Derivatives**

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(1) This question paper contains two parts A and B. Each part is compulsory. Part A carries 24 marks and Part B carries 56 marks.
(2) Part-A has Objective Type Questions and Part -B has Descriptive Type Questions.
(3) Both Part A and Part B have internal choices.

Part – A:

(1) It consists of two sections-I and II.
(2) Section I comprises of 16 very short answer type questions.
(3) Section II contains 2 case studies. Each case study comprises of 5 case-based MCQs. An examinee is to attempt any 4 out of 5 MCQs.

Part – B:

(1) It consists of three sections-III, IV and V.
(2) Section III comprises of 10 questions of 2 marks each.
(3) Section IV comprises of 7 questions of 3 marks each.
(4) Section V comprises of 3 questions of 5 marks each.
(5) Internal choice is provided in 3 questions of Section-III, 2 questions of Section-IV and 3 questions of Section-V. You have to attempt only one of the alternatives in all such questions.

PART-A

SECTION-I

All questions are compulsory. In case of internal choices attempt any one.

1. If a point lies on the y-axis, then what are its x-coordinate and z-coordinate?

2. Find the value of $\lim\limits_{x\to 0}\dfrac{\tan x}{x}$.

OR

Find the value of $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 2x}$.

3. What is the locus of a point for which $y = 0, z = 0$?

4. Find $f'(x)$, if $f(x) = x \sin x$.

5. Find the point on x-axis which is equidistant from the point $A(3, 2, 2)$ and $B(5, 5, 4)$.

6. Find the distance of the point $A(2, 3, 4)$ from the y-axis.

OR

Find the distance between the points $P(1, -3, 4)$ and $Q(-4, 1, 2)$.

7. Find the derivative of $x^3 \sec x$ w.r.t. x.

8. Find $\lim\limits_{x\to 2}\dfrac{x^5-32}{x-2}$.

OR

Evaluate : $\lim\limits_{x\to -1}\dfrac{x^{10}+x^5+1}{x-1}$

9. Find the point on y-axis which is at a distance of $\sqrt{10}$ units from the point (1, 2, 3).

10. Find the derivative of $(x^2 + 1)\cos x$.

11. Find the derivative of $f(x) = \sin x + \cos x$ using first principle.

12. Evaluate : $\lim\limits_{x\to -\infty}\dfrac{2x-1}{\sqrt{x^2+2x+1}}$

OR

Evaluate : $\lim\limits_{x\to 2}\left[\dfrac{x^3-2x^2}{x^2-5x+6}\right]$

13. Find the value of $\lim\limits_{x\to 1}\dfrac{x^{1/3}-1}{x^{1/6}-1}$.

14. Find the derivative of $f(x) = \dfrac{2x+3}{x-2}$ using first principle.

15. Find the perpendicular distance of the point (6, 5, 8) from z-axis.

16. Differentiate $\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^2$ w.r.t. x.

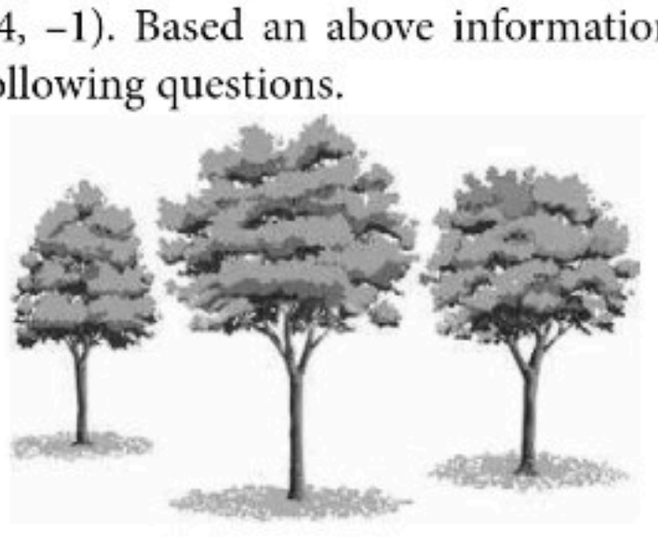
OR

Find the derivative of function
$y = 1 + x + x^2 + x^3 \ldots\ldots + x^{50}$ at $x = 1$.

SECTION - II

Both the case study based questions are compulsory. Attempt any 4 sub-parts from each question 17 and 18. Each sub-part carries 1 mark.

17. Rashi planted three trees in her garden. She planted the trees in such a way that the line joining the three trees makes a triangle. The coordinates of the three points (representing the trees) are $A(2, 3, 4)$, $B(4, -2, 6)$, $C(5, 4, -1)$. Based an above information, answer the following questions.



(i) Find the distance between tree A and tree B.

(a) 5 units (b) $\sqrt{45}$ units

(c) $\sqrt{33}$ units (d) 6 units

(ii) Find the distance between tree B and tree C.

(a) $\sqrt{83}$ units (b) $\sqrt{86}$ units

(c) $\sqrt{35}$ units (d) $\sqrt{77}$ units

(iii) The three points represents a/an _____.

(a) scalene triangle (b) right angled triangle

(c) isosceles triangle (d) equilateral triangle

(iv) Find the ratio in which a point (10, 0, 26) divides the line segment joining the tree A and tree B.

(a) 3 : 2 (b) 4 : 5 (c) 2 : 3 (d) 1 : 3

(v) Find the mid point of line segment joining the tree B and tree C.

(a) $\left(\dfrac{3}{2}, 2, \dfrac{3}{2}\right)$ (b) (1, 2, 3)

(c) (2, 3, 5) (d) $\left(\dfrac{9}{2}, 1, \dfrac{5}{2}\right)$

18. A function f is said to be a polynomial function if $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where a_i's are real numbers $n \in N$. Then, $\lim\limits_{x\to a} f(x) = a_0 + a_1a + a_2a^2 + \ldots + a_na^n = f(a)$.

A function f is said to be a rational function, if $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials such that $h(x) \neq 0$. Then

$$\lim_{x\to a} f(x) = \lim_{x\to a}\frac{g(x)}{h(x)} = \frac{\lim\limits_{x\to a} g(x)}{\lim\limits_{x\to a} h(x)} = \frac{g(a)}{h(a)}.$$

Note : $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a} = na^{n-1}$, where n is any rational number and a is positive.

Based on the above information, answer the following questions.

(i) $\lim\limits_{x\to 1}\dfrac{x+x^2+\ldots+x^n-n}{x-1}$ is

(a) n (b) $\dfrac{n+1}{2}$

(c) $\dfrac{n(n+1)}{2}$ (d) $\dfrac{n(n-1)}{2}$

(ii) $\lim\limits_{x\to -1}[1+x+x^2+\ldots+x^{10}]$ is

(a) 0 (b) 1

(c) 2 (d) does not exist

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(iii) $\lim_{x\to 2}\left(\frac{x^2-3x+2}{x^2+x-6}\right)=$

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) 1

(iv) The value of $\lim_{x\to 1}\frac{x^{1/4}-x^{1/5}}{x^3-1}$ is

(a) $\frac{1}{20}$ (b) $\frac{1}{40}$ (c) $\frac{1}{60}$ (d) $\frac{3}{20}$

(v) $\lim_{x\to 0}\left(\frac{\sqrt{1+x}-1}{x}\right)=$

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{5}{2}$

PART-B

SECTION-III

All questions are compulsory. In case of internal choices attempt any one.

19. Find the mid-point of the line joining the points (– 1, 2, 3) and (2, – 1, 3).

20. Evaluate : $\lim_{x\to\infty}\left(\frac{x^3}{3x^2-4}-\frac{x^2}{3x+2}\right)$

OR

Evaluate : $\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$

21. If $y=x^2+\sin x+\frac{1}{x^2}$, then find $\frac{dy}{dx}$.

22. Find the ratio in which the line segment joining the points (2, 4, 5) and (3, 5, –4) is divided by the xz-plane.

23. Find the derivative of $\frac{\cos x}{1+\sin x}$.

OR

If $x\sqrt{1+y}+y\sqrt{1+x}=0$, then find $\frac{dy}{dx}$.

24. Find the derivative of $x^5(3-6x^{-9})$ with respect to x.

25. Find the third vertex of triangle whose centroid is origin and two vertices are (2, 4, 6) and (0, –2, –5).

26. If $f(x)=\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2$, then find $f'\left(\frac{\pi}{4}\right)$.

27. Verify that (3, –2, 4), (1, 0, –2), (–1, 2, –8) are collinear.

OR

Find the locus of the point which is equidistant from the points $A(0, 2, 3)$ and $B(2, -2, 1)$.

28. Evaluate : $\lim_{x\to a}\frac{x^{5/2}-a^{5/2}}{\sqrt{x}-\sqrt{a}}$.

SECTION-IV

All questions are compulsory. In case of internal choices attempt any one.

29. For the function $f(x)=\begin{cases}a+bx, & x<1\\ 4, & x=1\\ b-ax, & x>1\end{cases}$

$\lim_{x\to 1}f(x)=f(1)$. Find the possible values of a and b.

30. Find the coordinates of the point which is three-fifth of the way from (3, 4, 5) to (–2, –1, 0).

31. Find the derivative of $f(x)=2x^2+3x-5$ at $x=-1$ using first principle. Also, show that $f'(0)+3f'(-1)=0$.

OR

Find the derivative of $f(x)=\sqrt{\cos x}$ using first principle.

32. Find the ratio in which the line segment joining the points (2, 4, –3) and (–3, 5, 4) is divided by the XY-plane. Also, find the coordinates of point of division.

33. Find the value of $\lim_{x\to 0}\frac{(1-\cos 2x)\sin 5x}{x^2\sin 3x}$.

34. Let $A(3, 2, 0)$, $B(5, 3, -2)$ and $C(-9, 6, -3)$ be three points forming a triangle. The bisector AD of $\angle BAC$ meets side BC on D. Find the coordinates of D.

OR

Find the lengths of the medians of the triangle with vertices $A(0, 0, 8)$, $B(0, 6, 0)$ and $C(6, 0, 0)$.

35. If $y=(x+1)(x+2)(x+3)(x+4)(x+5)$, then find the value of $\frac{dy}{dx}$ at $x=0$.

SECTION-V

All questions are compulsory. In case of internal choices attempt any one.

36. $ABCD$ is a parallelogram whose 3 consecutive vertices are $A(1, 2, 3)$, $B(-1, -2, -1)$ and $C(2, 3, 2)$. Find the fourth vertex D. Check whether $ABCD$ is a rectangle or not.

OR

The mid-points of the sides of a triangle are (1, 5, –1), (0, 4, –2) and (2, 3, 4). Find its vertices.

37. If $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$

For what integers m and n does both $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 1} f(x)$ exist?

OR

If $\lim\limits_{x\to 0} f(x)$ exists, then find the value of K where

$$f(x)=\begin{cases} \dfrac{\sqrt{1+Kx}-\sqrt{1-Kx}}{5x}, & x<0 \\ \dfrac{x\cos x+\sin x}{x^2+\tan x}, & x\ge 0 \end{cases}$$

38. (i) Find the derivative of cot $(3x - 2)$ using first principle method.

(ii) Find the derivative of $f(x)=\dfrac{x}{1+\tan x}$.

OR

Find the derivative of cosec x with respect to x, from first principle.

SOLUTIONS

1. If a point lies on y-axis, then its x-coordinate is 0 and z-coordinate is 0.

2. We know $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=1$

OR

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 2x}=\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\times\frac{2x}{\sin 2x}\times\frac{3}{2}\right)$$

$$=\frac{3}{2}\cdot\lim_{3x\to 0}\frac{\sin 3x}{3x}\times\lim_{2x\to 0}\frac{2x}{\sin 2x}$$

$(\because$ as $x\to 0$, $3x\to 0$ and $2x\to 0)$

$$=\frac{3}{2}\times 1\times 1=\frac{3}{2}$$

3. Locus of the point for which $y = 0$, $z = 0$ is x-axis, since on x-axis both $y = 0$ and $z = 0$.

4. We have, $f(x) = x \sin x$

Differentiating both sides, we get

$$f'(x)=\frac{d}{dx}(x\sin x)$$

$$=x\cdot\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(x)=x\cos x+\sin x$$

5. The point on the x-axis is of the form $P(x, 0, 0)$. Since the points A and B are equidistant from P. Therefore $PA^2 = PB^2$

$$\Rightarrow\quad (x-3)^2+(0-2)^2+(0-2)^2=(x-5)^2+(0-5)^2+(0-4)^2$$

$$\Rightarrow\quad 4x=25+25+16-17\Rightarrow x=\frac{49}{4}$$

Thus, the point P on the x-axis is $\left(\dfrac{49}{4},0,0\right)$ which is equidistant from A and B.

6. The coordinates of foot of perpendicular from (2, 3, 4) on y-axis are (0, 3, 0)

$$\therefore\quad \text{Required distance}=\sqrt{(0-2)^2+(3-3)^2+(0-4)^2}$$

$$=\sqrt{4+0+16}=\sqrt{20}=2\sqrt{5}\text{ units.}$$

OR

The distance between the points $P(1, -3, 4)$ and $Q(-4, 1, 2)$ is

$$PQ=\sqrt{(-4-1)^2+(1+3)^2+(2-4)^2}$$

$$\left[\because\ PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\right]$$

$$=\sqrt{25+16+4}=\sqrt{45}=3\sqrt{5}\text{ units}$$

7. $\dfrac{d}{dx}(x^3\sec x)=x^3\cdot\dfrac{d}{dx}(\sec x)+\sec x\cdot\dfrac{d}{dx}(x^3)$

$$=x^3\cdot\sec x\tan x+\sec x\cdot 3x^2=x^2\sec x\,(x\tan x+3)$$

8. We have, $\lim\limits_{x\to 2}\dfrac{x^5-32}{x-2}=\lim\limits_{x\to 2}\left(\dfrac{x^5-2^5}{x-2}\right)$

$$=5\times 2^{5-1}\qquad\left[\because\ \lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)=na^{n-1}\right]$$

$$=5\times 2^4=5\times 16=80.$$

OR

$$\lim_{x\to -1}\frac{x^{10}+x^5+1}{x-1}=\frac{1-1+1}{-1-1}=-\frac{1}{2}$$

9. Let the point P be on y-axis. Therefore, it is of the form $P(0, y, 0)$.

The point (1, 2, 3) is at a distance $\sqrt{10}$ units from $(0, y, 0)$.

$$\Rightarrow\quad \sqrt{(1-0)^2+(2-y)^2+(3-0)^2}=\sqrt{10}$$

$$\Rightarrow\quad y^2-4y+4=0\Rightarrow (y-2)^2=0\Rightarrow y=2$$

Hence the required point is (0, 2, 0).

10. Let $f(x) = (x^2 + 1) \cos x$

Differentiating both sides, we get

$$\therefore\quad f'(x)=(x^2+1)'\cos x+(\cos x)'(x^2+1)$$

$$=(2x+0)\cos x+(-\sin x)(x^2+1)$$

$$=2x\cos x-(x^2+1)\sin x$$

11. We have $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{\sin(x+h)+\cos(x+h)-\sin x-\cos x}{h}$$

$$= \lim_{h\to 0} \frac{\begin{matrix}\sin x\cos h+\cos x\sin h+\cos x\cos h-\sin x\sin h\\ -\sin x-\cos x\end{matrix}}{h}$$

$$= \lim_{h\to 0} \frac{\sin h\,(\cos x-\sin x)+\sin x(\cos h-1)+\cos x(\cos h-1)}{h}$$

$$= \lim_{h\to 0} \frac{\sin h}{h}(\cos x-\sin x)+\lim_{h\to 0}\sin x\frac{(\cos h-1)}{h} + \lim_{h\to 0}\cos x\frac{(\cos h-1)}{h}$$

$= (\cos x - \sin x) + 0 + 0 = \cos x - \sin x$

12. Put $x = -1/y$ as $x \to -\infty$ so that $y \to 0^+$

$$\therefore \quad \lim_{x\to-\infty} \frac{2x-1}{\sqrt{x^2+2x+1}}$$

$$= \lim_{y\to 0^+} \frac{\frac{-2}{y}-1}{\sqrt{\frac{1}{y^2}-\frac{2}{y}+1}} = \lim_{y\to 0^+} \frac{-2-y}{\sqrt{1-2y+y^2}} = -2.$$

OR

We have $\lim_{x\to 2} \frac{x^3-2x^2}{x^2-5x+6} = \lim_{x\to 2} \frac{x^2(x-2)}{(x-2)(x-3)}$

$$= \lim_{x\to 2} \frac{x^2}{(x-3)} = \frac{(2)^2}{2-3} = \frac{4}{-1} = -4$$

13. Consider, $\lim_{x\to 1} \frac{x^{1/3}-1}{x^{1/6}-1} = \lim_{x\to 1} \frac{(x^{1/6})^2-1}{x^{1/6}-1}$

$$= \lim_{x\to 1}(x^{1/6}+1) = 1+1 = 2$$

14. We have, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{\frac{2(x+h)+3}{x+h-2}-\frac{2x+3}{x-2}}{h}$$

$$= \lim_{h\to 0} \frac{(2x+2h+3)(x-2)-(2x+3)(x+h-2)}{h(x-2)(x+h-2)}$$

$$= \lim_{h\to 0} \frac{(2x+3)(x-2)+2h(x-2)-(2x+3)(x-2)-h(2x+3)}{h(x-2)(x+h-2)}$$

$$= -\frac{7}{(x-2)^2}$$

15. Perpendicular distance of the point (6, 5, 8) from z-axis $= \sqrt{6^2+5^2+0^2} = \sqrt{36+25} = \sqrt{61}$ units.

16. We have, $\frac{d}{dx}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$

$$= \frac{d}{dx}\left(x+\frac{1}{x}+2\right) = \frac{d}{dx}(x)+\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{d}{dx}(2)$$

$$= 1+\left(\frac{-1}{x^2}\right)+0 = 1-\frac{1}{x^2}$$

OR

We have, $y = 1 + x + x^2 + \ldots\ldots + x^{50}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(1)+\frac{d}{dx}(x)+\frac{d}{dx}(x^2)+\ldots\ldots+\frac{d}{dx}(x^{50})$$

$= 0 + 1 + 2x + \ldots\ldots + 50x^{49}$ ...(i)

Putting $x = 1$ in (i), we get

$$\frac{dy}{dx} = 1+2+3+\ldots\ldots+50 = \frac{50\times 51}{2} = 1275.$$

17. **(i) (c) :** Distance between tree A and tree B

$$= \sqrt{(4-2)^2+(-2-3)^2+(6-4)^2}$$

$$= \sqrt{4+25+4} = \sqrt{33} \text{ units}$$

(ii) (b) : Distance between tree B and tree C

$$= \sqrt{(5-4)^2+(4+2)^2+(-1-6)^2}$$

$$= \sqrt{1+36+49} = \sqrt{86} \text{ units}$$

(iii) (a) : As, $AB = \sqrt{33}$ units, $BC = \sqrt{86}$ units

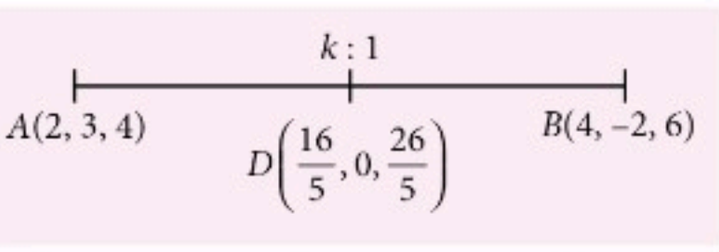
Now, $AC = \sqrt{(5-2)^2+(4-3)^2+(-1-4)^2}$

$$= \sqrt{9+1+25} = \sqrt{35} \text{ units}$$

$\because \quad AB \neq BC \neq AC$

$\therefore \quad \Delta ABC$ is a scalene triangle.

(iv) (a) : Let the point $D\left(\frac{16}{5}, 0, \frac{26}{5}\right)$ divides the line segment AB in the ratio $k : 1$



So, $\frac{16}{5} = \frac{4k+2}{k+1}$; $0 = \frac{-2k+3}{k+1}$; $\frac{26}{5} = \frac{6k+4}{k+1}$

$\Rightarrow \quad 16k + 16 = 20k + 10$

$\Rightarrow \quad 4k = 6 \Rightarrow \quad k = \frac{3}{2}$

So, required ratio is 3 : 2.

(v) (d) : Required mid point is

$$\left(\frac{4+5}{2}, \frac{-2+4}{2}, \frac{6-1}{2}\right) \text{ i.e., } \left(\frac{9}{2}, 1, \frac{5}{2}\right)$$

18. **(i) (c) :** We have

$$\lim_{x\to 1}\left\{\frac{x+x^2+\ldots+x^n-n}{x-1}\right\}$$

$$= \lim_{x\to 1}\left\{\frac{x+x^2+\ldots+x^n-1-1-1-\ldots-1}{x-1}\right\}$$

$$= \lim_{x\to 1}\left\{\frac{(x-1)+(x^2-1)+(x^3-1)+\ldots+(x^n-1)}{x-1}\right\}$$

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$= \lim_{x\to 1}\{1+(x+1)+(x^2+x+1)+...\}$

$= 1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$

(ii) (b) : $\lim_{x\to -1}[1+x+x^2+...+x^{10}]$

$= 1 + (-1) + (-1)^2 + ... + (-1)^{10} = 1 - 1 + 1 - ... + 1 = 1$

(iii) (a) : We have, $\lim_{x\to 2}\left(\dfrac{x^2-3x+2}{x^2+x-6}\right)$

$= \lim_{x\to 2}\dfrac{(x-2)(x-1)}{(x+3)(x-2)} = \lim_{x\to 2}\left(\dfrac{x-1}{x+3}\right) = \dfrac{2-1}{2+3} = \dfrac{1}{5}.$

(iv) (c) : We have, $\lim_{x\to 1}\dfrac{x^{1/4}-x^{1/5}}{x^3-1} = \lim_{x\to 1}\dfrac{(x^{1/4}-1)-(x^{1/5}-1)}{x^3-1}$

$= \lim_{x\to 1}\left(\dfrac{\dfrac{x^{1/4}-1^{1/4}}{x-1}}{\dfrac{x^3-1^3}{x-1}} - \dfrac{\dfrac{x^{1/5}-1^{1/5}}{x-1}}{\dfrac{x^3-1^3}{x-1}}\right) = \dfrac{1}{12} - \dfrac{1}{15} = \dfrac{1}{60}$

(v) (b) : Put $y = 1 + x$, so that $y \to 1$ when $x \to 0$.

Now, $\lim_{x\to 0}\dfrac{\sqrt{1+x}-1}{x} = \lim_{y\to 1}\dfrac{\sqrt{y}-1}{y-1} = \lim_{y\to 1}\dfrac{y^{\frac{1}{2}}-1^{\frac{1}{2}}}{y-1}$

$= \dfrac{1}{2}(1)^{\frac{1}{2}-1} = \dfrac{1}{2}$

19. Let $P(x, y, z)$ be the mid point of the line joining the points $(-1, 2, 3)$ and $(2, -1, 3)$.

$\therefore \quad P(x, y, z) = \left(\dfrac{-1+2}{2}, \dfrac{2-1}{2}, \dfrac{3+3}{2}\right) = \left(\dfrac{1}{2}, \dfrac{1}{2}, 3\right).$

20. We have, $\lim_{x\to\infty}\left(\dfrac{x^3}{3x^2-4} - \dfrac{x^2}{3x+2}\right)$

$= \lim_{x\to\infty}\left(\dfrac{x^3(3x+2)-x^2(3x^2-4)}{(3x^2-4)(3x+2)}\right)$

$= \lim_{x\to\infty}\left(\dfrac{3x^4+2x^3-3x^4+4x^2}{9x^3+6x^2-12x-8}\right) = \lim_{x\to\infty}\left(\dfrac{2x^3+4x^2}{9x^3+6x^2-12x-8}\right)$

$= \lim_{x\to\infty}\left(\dfrac{2+\dfrac{4}{x}}{9+\dfrac{6}{x}-\dfrac{12}{x^2}-\dfrac{8}{x^3}}\right) = \dfrac{2}{9}$

OR

Put $y = 2 + x$ so that when $x \to 0$, $y \to 2$.

Now, $\lim_{x\to 0}\dfrac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{y\to 2}\dfrac{y^{\frac{1}{2}}-2^{\frac{1}{2}}}{y-2}$

$= \dfrac{1}{2}(2)^{\frac{1}{2}-1} = \dfrac{1}{2}\cdot 2^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{2}}$

21. We have, $y = x^2 + \sin x + \dfrac{1}{x^2}$

Differentiating w.r.t. x both sides, we get

$\dfrac{dy}{dx} = \dfrac{d}{dx}(x^2) + \dfrac{d}{dx}(\sin x) + \dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)$

$\Rightarrow \quad \dfrac{dy}{dx} = 2x + \cos x + (-2)x^{-3} = 2x + \cos x - \dfrac{2}{x^3}$

22. Let the join of $P(2, 4, 5)$ and $Q(3, 5, -4)$ be divided by xz-plane in the ratio $k : 1$ at the point $R(x, y, z)$.

Therefore $x = \dfrac{3k+2}{k+1}$, $y = \dfrac{5k+4}{k+1}$, $z = \dfrac{-4k+5}{k+1}$

Since the point $R(x, y, z)$ lies on xz-plane, so its y-coordinate is zero, *i.e.*, $\dfrac{5k+4}{k+1} = 0 \Rightarrow k = -\dfrac{4}{5}$

Hence, the required ratio is 4 : 5 externally.

23. Let $y = \dfrac{\cos x}{1+\sin x}$.

Differentiating both sides with respect to x, we get

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\cos x}{1+\sin x}\right)$

$= \dfrac{(1+\sin x)\dfrac{d}{dx}(\cos x) - \cos x\dfrac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$

$= \dfrac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$

$= \dfrac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} = \dfrac{-(1+\sin x)}{(1+\sin x)^2} = \dfrac{-1}{1+\sin x}$

OR

Given $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow \quad x\sqrt{1+y} = -y\sqrt{1+x}$ ...(i)

Squaring both the sides, we get

$x^2(1 + y) = y^2(1 + x) \Rightarrow x^2 - y^2 + x^2y - xy^2 = 0$

$\Rightarrow \quad (x - y)(x + y + xy) = 0$

$\Rightarrow \quad y = x$ or $y(1 + x) = -x$

$\Rightarrow \quad y = x$ or $y = -\dfrac{x}{1+x}$

Note that $y = x$ does not satisfy (i) in general.

$\Rightarrow \quad y = -\dfrac{x}{1+x} \Rightarrow \dfrac{dy}{dx} = -\dfrac{(1+x)\cdot 1 - x}{(1+x)^2} = \dfrac{-1}{(1+x)^2}$

Monthly Test Drive-7 CLASS XII ANSWER KEY

1.	(d)	**2.**	(b)	**3.**	(a)	**4.**	(c)	**5.**	(d)
6.	(d)	**7.**	(c)	**8.**	(a,c,d)	**9.**	(b,d)	**10.**	(a,c,d)
11.	(a,c)	**12.**	(a,c)	**13.**	(b,c,d)	**14.**	(b)	**15.**	(d)
16.	(a)	**17.**	(6)	**18.**	(9997)	**19.**	(314)	**20.**	(3)

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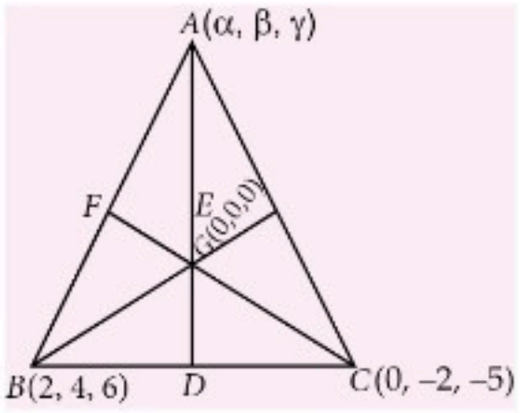
24. $\frac{d}{dx}\{x^5(3-6\cdot x^{-9})\} = \frac{d}{dx}\{3\cdot x^5 - 6x^{-4}\}$

$= \frac{d}{dx}(3x^5) - \frac{d}{dx}(6x^{-4})$

$= 3\,(5x^4) - 6\,(-4)x^{-4-1}$

$= 15x^4 + 24\,x^{-5} = 15x^4 + \frac{24}{x^5}$

25. Let third vertex of ΔABC be $A(\alpha, \beta, \gamma)$



Since the coordinates of centroid G are $(0, 0, 0)$

$\Rightarrow \quad 0 = \frac{\alpha+2+0}{3} \Rightarrow \alpha = -2$

$\Rightarrow \quad 0 = \frac{\beta+4-2}{3} \Rightarrow \beta = -2$

and $0 = \frac{\gamma+6-5}{3} \Rightarrow \gamma = -1$

So, the third vertex of the triangle is $A(-2, -2, -1)$.

26. $f(x) = \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2$

$= \cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}$

$\Rightarrow \quad f(x) = 1 + \sin x \Rightarrow f'(x) = 0 + \cos x = \cos x$

$\therefore \quad f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

27. Let $A(3, -2, 4)$, $B(1, 0, -2)$ and $C(-1, 2, -8)$ be given points.

$\therefore$ By distance formula,

$AB = \sqrt{(1-3)^2 + (0+2)^2 + (-2-4)^2}$

$= \sqrt{4+4+36} = \sqrt{44}$ units

$BC = \sqrt{(-1-1)^2 + (2-0)^2 + (-8+2)^2}$

$= \sqrt{4+4+36} = \sqrt{44}$ units

$AC = \sqrt{(-1-3)^2 + (2+2)^2 + (-8-4)^2}$

$= \sqrt{16+16+144} = \sqrt{176} = 2\sqrt{44}$ units

$\therefore \quad AB + BC = 2\sqrt{44}$ units $= AC$

Thus, the given points are collinear.

OR

Let $P(x, y, z)$ be any point which is equidistant from $A(0, 2, 3)$ and $B(2, -2, 1)$.

Then, $PA = PB \Rightarrow PA^2 = PB^2$

$\Rightarrow \quad (x-0)^2 + (y-2)^2 + (z-3)^2$

$= (x-2)^2 + (y+2)^2 + (z-1)^2$

$\Rightarrow \quad 4x - 8y - 4z + 4 = 0 \Rightarrow x - 2y - z + 1 = 0$

Hence, the required locus is $x - 2y - z + 1 = 0$

28. We have, $\lim\limits_{x\to a}\frac{x^{5/2}-a^{5/2}}{x^{1/2}-a^{1/2}} = \lim\limits_{x\to a}\frac{\frac{x^{5/2}-a^{5/2}}{x-a}}{\frac{x^{1/2}-a^{1/2}}{x-a}}$

$= \frac{\frac{5}{2}\cdot a^{3/2}}{\frac{1}{2}\cdot a^{-1/2}} = 5a^2$

29. We have, $f(x) = \begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax, & x>1 \end{cases}$

Given, $\lim\limits_{x\to 1} f(x) = f(1)$

$\Rightarrow \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = f(1)$

$\Rightarrow \lim\limits_{h\to 0} f(1-h) = \lim\limits_{h\to 0} f(1+h) = f(1)$

$\Rightarrow \lim\limits_{h\to 0}\{a+b(1-h)\} = \lim\limits_{h\to 0}\{b-a(1+h)\} = 4$

$\Rightarrow \quad a + b = b - a = 4 \Rightarrow a = 0, b = 4$

30. Let $A = (3, 4, 5)$, $B = (-2, -1, 0)$ and $P(x, y, z)$ be the required point. As P is three-fifth of the way from A to B. So, P divides $[AB]$ in the ratio 3 : 2.

$\therefore \quad P \equiv \left(\frac{3\times(-2)+2\times 3}{3+2}, \frac{3\times(-1)+2\times 4}{3+2}, \frac{3\times 0+2\times 5}{3+2}\right)$

$\Rightarrow \quad P = (0, 1, 2)$.

31. Let $f(x) = 2x^2 + 3x - 5$

$\therefore$ Using first principle,

$f'(x) = \lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim\limits_{h\to 0}\frac{[2(x+h)^2 + 3(x+h) - 5] - [2x^2 + 3x - 5]}{h}$

$= \lim\limits_{h\to 0}\frac{2[(x+h)^2 - x^2] + 3(x+h-x)}{h}$

$= \lim\limits_{h\to 0}\frac{2[x^2 + h^2 + 2xh - x^2] + 3h}{h}$

$= \lim\limits_{h\to 0}\frac{2h^2 + 4xh + 3h}{h} = \lim\limits_{h\to 0} 2h + 4x + 3 = 4x + 3$

$\therefore \quad f'(x) = 4x + 3$

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$\Rightarrow \quad f'(-1) = 4(-1) + 3 = -1$ and $f'(0) = 3$

Now, $f'(0) + 3f'(-1) = 3 + 3(-1) = 0$.

OR

Given, $f(x) = \sqrt{\cos x}$

From first principle, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{\cos(x+h)} - \sqrt{\cos x}}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h[\sqrt{\cos(x+h)} + \sqrt{\cos x}]}$$

$$= \lim_{h\to 0} \frac{-2\sin\left(x + \frac{h}{2}\right).\sin\frac{h}{2}}{h[\sqrt{\cos(x+h)} + \sqrt{\cos x}]}$$

$$= \lim_{h\to 0} \frac{-2\sin\left(x + \frac{h}{2}\right)}{[\sqrt{\cos(x+h)} + \sqrt{\cos x}]} \cdot \frac{\sin\frac{h}{2}}{\frac{h}{2}\cdot 2}$$

$$= \frac{-\sin x}{\sqrt{\cos x} + \sqrt{\cos x}} \times 1 = \frac{-\sin x}{2\sqrt{\cos x}}$$

32. Let the line segment joining the points $P(2, 4, -3)$ and $Q(-3, 5, 4)$ is divided by the XY-plane at the point R in the ratio $k : 1$ internally.

$\therefore$ Coordinates of point R are

$$\left(\frac{-3k+2}{k+1}, \frac{5k+4}{k+1}, \frac{4k-3}{k+1}\right)$$

Since the point R lies on XY-plane

$\therefore \quad z$ coordinate $= 0$

$$\Rightarrow \quad \frac{4k-3}{k+1} = 0 \Rightarrow k = \frac{3}{4}$$

Thus, the required ratio is 3 : 4.

Also, the coordinates of $R = \left(\frac{-3\times\frac{3}{4}+2}{\frac{3}{4}+1}, \frac{5\times\frac{3}{4}+4}{\frac{3}{4}+1}, 0\right)$

$$= \left(\frac{-1}{7}, \frac{31}{7}, 0\right)$$

33. $\lim_{x\to 0} \frac{(1-\cos 2x)\sin 5x}{x^2 \sin 3x} = \lim_{x\to 0} \frac{2\sin^2 x \sin 5x}{x^2 \sin 3x}$

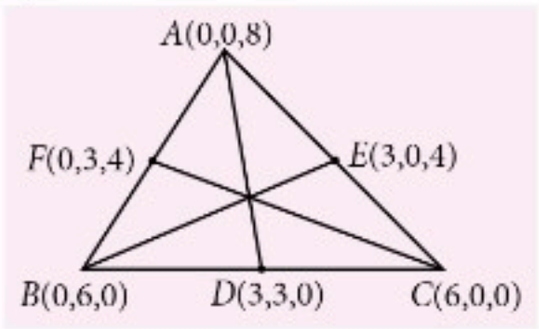
$$= \lim_{x\to 0} \left(\frac{2\sin^2 x}{x^2}\right) \frac{\left(\frac{\sin 5x}{5x}\right)\times 5}{\left(\frac{\sin 3x}{3x}\right)\times 3}$$

$$= 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \times \frac{5\lim\limits_{5x\to 0}\left(\frac{\sin 5x}{5x}\right)}{3\lim\limits_{3x\to 0}\left(\frac{\sin 3x}{3x}\right)} = \frac{2\times 5}{3} = \frac{10}{3}$$

34. *Refer to Answer 58, page no. 274 of MTG CBSE Champion Mathematics, Class 11.*

OR

Let D, E and F be the mid-points of sides BC, CA and AB respectively.



The coordinates of D, E and F are

$$D\left(\frac{0+6}{2}, \frac{6+0}{2}, \frac{0+0}{2}\right) = (3,3,0)$$

$$E\left(\frac{6+0}{2}, \frac{0+0}{2}, \frac{0+8}{2}\right) = (3,0,4)$$

and $F\left(\frac{0+0}{2}, \frac{0+6}{2}, \frac{8+0}{2}\right) = (0,3,4)$ respectively.

$\therefore \quad AD = \sqrt{(0-3)^2 + (0-3)^2 + (8-0)^2}$

$= \sqrt{9+9+64} = \sqrt{82}$ units

$BE = \sqrt{(0-3)^2 + (6-0)^2 + (0-4)^2}$

$= \sqrt{9+36+16} = \sqrt{61}$ units

and, $CF = \sqrt{(6-0)^2 + (0-3)^2 + (0-4)^2}$

$= \sqrt{36+9+16} = \sqrt{61}$ units

35. We have, $y = (x+1)(x+2)(x+3)(x+4)(x+5)$

$$\Rightarrow \quad \frac{dy}{dx} = (x+2)(x+3)(x+4)(x+5) + (x+1)(x+3)(x+4)(x+5) + \ldots..$$

(Using product rule)

$$\therefore \left(\frac{dy}{dx}\right)_{x=0} = 2\cdot3\cdot4\cdot5 + 1\cdot3\cdot4\cdot5 + 1\cdot2\cdot4\cdot5 + 1\cdot2\cdot3\cdot5 + 1\cdot2\cdot3\cdot4$$

$$= 120 + 60 + 40 + 30 + 24 = 274.$$

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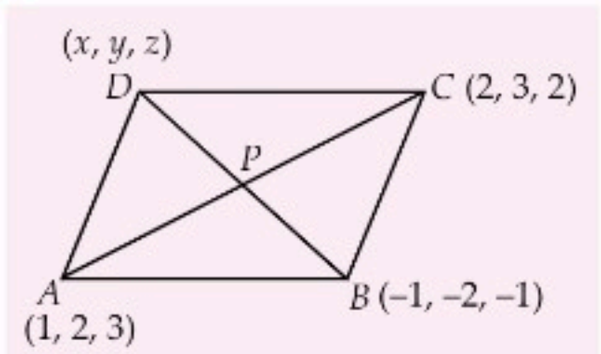
36. Let the fourth vertex of the parallelogram $ABCD$ be $D\ (x, y, z)$.



Then by using mid-point formula, the mid-point of AC is

$$P\left(\frac{1+2}{2}, \frac{2+3}{2}, \frac{3+2}{2}\right) \text{ i.e. } P\left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)$$

Also, using mid-point formula for line BD, we get

$$P\left(\frac{-1+x}{2}, \frac{-2+y}{2}, \frac{-1+z}{2}\right)$$

Now, $\frac{3}{2} = \frac{-1+x}{2} \Rightarrow x = 4$

$\frac{5}{2} = \frac{-2+y}{2} \Rightarrow y = 7$

$\frac{5}{2} = \frac{-1+z}{2} \Rightarrow z = 6$

Hence, the coordinates of fourth vertex D are $(4, 7, 6)$.

Now, $AB = \sqrt{(-1-1)^2 + (-2-2)^2 + (-1-3)^2}$

$= \sqrt{4+16+16} = \sqrt{36} = 6$ units

$BC = \sqrt{(2+1)^2 + (3+2)^2 + (2+1)^2}$

$= \sqrt{9+25+9} = \sqrt{43}$ units

$CD = \sqrt{(4-2)^2 + (7-3)^2 + (6-2)^2}$

$= \sqrt{4+16+16} = \sqrt{36} = 6$ units

$AD = \sqrt{(4-1)^2 + (7-2)^2 + (6-3)^2}$

$= \sqrt{9+25+9} = \sqrt{43}$ units

Clearly $AB = CD$ and $BC = AD$

$\Rightarrow$ Opposite sides are equal

Hence, it is a parallelogram.

Also, $AC = \sqrt{(2-1)^2 + (3-2)^2 + (2-3)^2}$

$= \sqrt{1+1+1} = \sqrt{3}$ units

$BD = \sqrt{(4+1)^2 + (7+2)^2 + (6+1)^2}$

$= \sqrt{25+81+49} = \sqrt{155}$ units

$\therefore \quad AC \neq BD$

$\Rightarrow$ Diagonals are not equal

$\therefore \quad ABCD$ is not a rectangle.

OR

Refer to Answer 72, page no. 276 of MTG CBSE Champion Mathematics, Class 11.

37. Consider function

$$f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \le x \le 1 \\ nx^3 + m, & x > 1 \end{cases}$$

For $\lim_{x \to 0} f(x)$ to exist, we must have,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

$\Rightarrow \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(0+h)$

$\Rightarrow \lim_{h \to 0} \{m(0-h)^2 + n\} = \lim_{h \to 0} \{n(0+h) + m\}$

$\Rightarrow n = m$

Thus, $\lim_{x \to 0} f(x)$ exists only when $n = m$.

For $\lim_{x \to 1} f(x)$ to exist, we must have,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$\Rightarrow \lim_{h \to 0} f(1-h) = \lim_{h \to 0} f(1+h)$

$\Rightarrow \lim_{h \to 0} \{n(1-h) + m\} = \lim_{h \to 0} \{n(1+h)^3 + m\}$

$\Rightarrow n + m = n + m$ which is true for all n and m.

OR

Given, $f(x) = \begin{cases} \dfrac{\sqrt{1+Kx} - \sqrt{1-Kx}}{5x}, & x < 0 \\ \dfrac{x\cos x + \sin x}{x^2 + \tan x}, & x \ge 0 \end{cases}$

L.H.L. $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0} \dfrac{\sqrt{1+Kx} - \sqrt{1-Kx}}{5x}$

$= \lim_{x \to 0} \dfrac{(1+Kx) - (1-Kx)}{5x\left(\sqrt{1+Kx} + \sqrt{1-Kx}\right)}$

$= \lim_{x \to 0} \dfrac{2Kx}{5x\left(\sqrt{1+Kx} + \sqrt{1-Kx}\right)}$

$= \dfrac{2K}{5\left(\sqrt{1} + \sqrt{1}\right)} = \dfrac{2K}{10} = \dfrac{K}{5}$

R.H.L. $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0} \dfrac{x\cos x + \sin x}{x^2 + \tan x}$

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$$= \lim_{x\to 0} \frac{\frac{x\cos x+\sin x}{x}}{\frac{x^2+\tan x}{x}} = \lim_{x\to 0} \frac{\cos x+\frac{\sin x}{x}}{x+\frac{\tan x}{x}} = \frac{1+1}{0+1} = \frac{2}{1} = 2$$

For $\lim_{x\to 0} f(x)$ to exist L.H.L. = R.H.L.

$$\Rightarrow \frac{K}{5} = 2 \Rightarrow K = 10$$

38. (i) Let $f(x) = \cot(3x - 2)$

Then, using first principle,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cot(3(x+h)-2)-\cot(3x-2)}{h}$$

$$= \lim_{h\to 0} \frac{\cot(3x+3h-2)-\cot(3x-2)}{h}$$

$$= \lim_{h\to 0} \frac{-\sin(3x+3h-2-3x+2)}{h[\sin(3x+3h-2)\sin(3x-2)]}$$

$$\left[\because \cot A - \cot B = \frac{\cos A}{\sin A} - \frac{\cos B}{\sin B} = \frac{\cos A\sin B - \sin A\cos B}{\sin A\sin B} = \frac{-\sin(A-B)}{\sin A\sin B}\right]$$

$$= \lim_{h\to 0} \frac{-\sin 3h\cdot(3)}{3h[\sin(3x+3h-2)\sin(3x-2)]}$$

$$= \lim_{h\to 0}\left(\frac{\sin 3h}{3h}\right)\cdot \lim_{h\to 0} \frac{-3}{[\sin(3x+3h-2)\sin(3x-2)]}$$

$$= \frac{-3}{\sin^2(3x-2)} = -3\operatorname{cosec}^2(3x-2)$$

(ii) Given, $f(x) = \frac{x}{1+\tan x}$

Now, $f'(x) = \frac{d}{dx}\left(\frac{x}{1+\tan x}\right)$

$$= \frac{(1+\tan x)\frac{d(x)}{dx} - x\cdot\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)-x\cdot\sec^2 x}{(1+\tan x)^2} = \frac{1+\tan x - x\sec^2 x}{(1+\tan x)^2}$$

OR

Refer to Answer 125, page no. 299 of MTG CBSE Champion Mathematics, Class 11.

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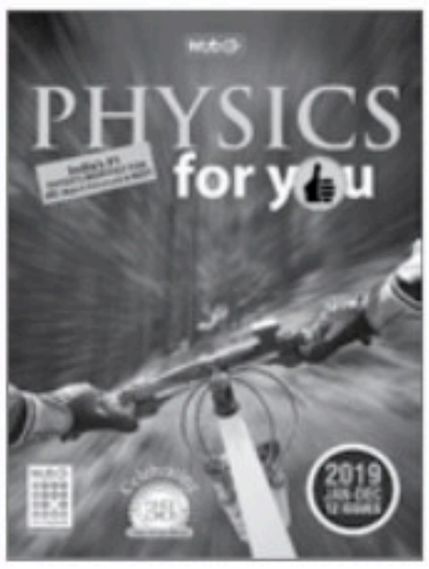
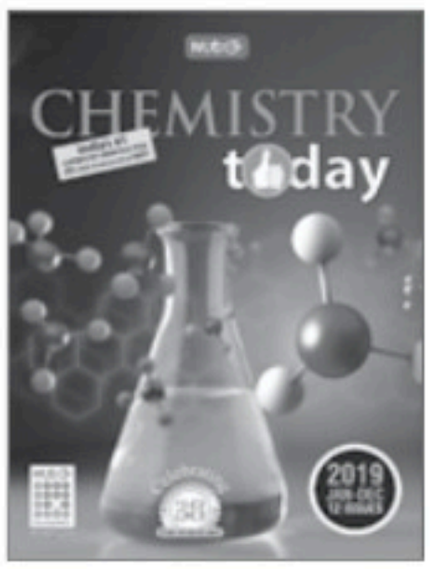
AVAILABLE BOUND VOLUMES





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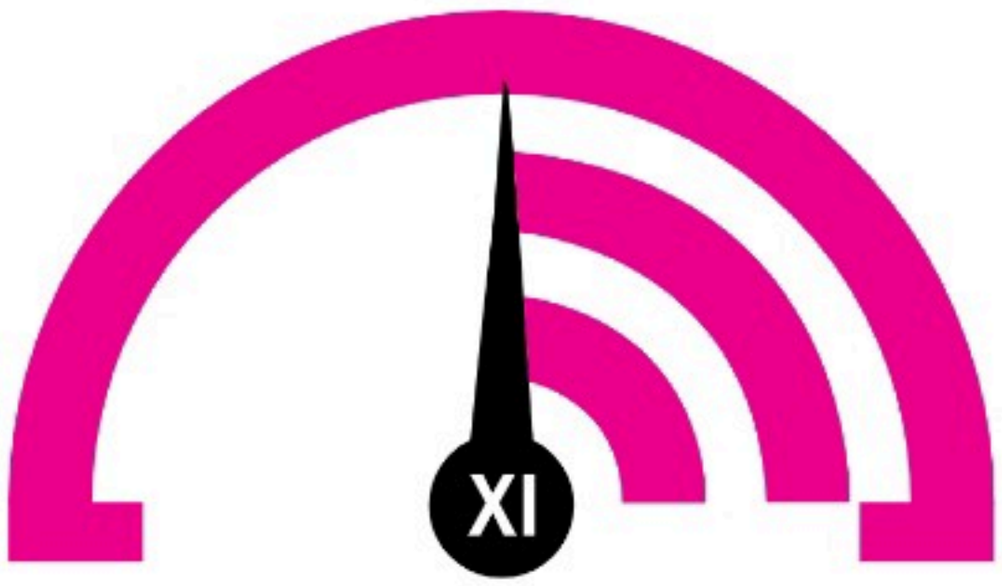
MONTHLY TEST DRIVE



This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80

Series 7 : Limits and Derivatives

Time Taken : 60 Min.

Only One Option Correct Type

1. The value of $\lim\limits_{x\to-\infty}(\sqrt{x^2-x+1}+x)$ is

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) –1

2. The value of $\lim\limits_{x\to0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to

(a) $-\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) 2

3. The value of $\lim\limits_{x\to0}\left\{\frac{a^x+b^x+c^x}{3}\right\}^{1/x}$ is

(a) abc (b) $(abc)^{1/3}$

(c) $\frac{1}{3}abc$ (d) none of these

4. If $y=|\cos x|+|\sin x|$, then $\frac{dy}{dx}$ at $x=\frac{2\pi}{3}$ is

(a) $\frac{1-\sqrt{3}}{2}$ (b) 0

(c) $\frac{\sqrt{3}-1}{2}$ (d) none of these

5. Let $f(x)=\frac{\log_e(x^2+e^x)}{\log_e(x^4+e^{2x})}$. If $\lim\limits_{x\to\infty}f(x)=l$ and $\lim\limits_{x\to-\infty}f(x)=m$, then

(a) $l=m$ (b) $l=2m$

(c) $2l=m$ (d) $l+m=0$

6. If $y=f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x)=\sin x^2$, then $\frac{dy}{dx}$ is equal to

(a) $\sin\left(\frac{2x-1}{x^2+1}\right)^2\left\{\frac{x^2+2x+2}{(x^2+1)^2}\right\}$

(b) $\sin\left(\frac{2x-1}{x^2+1}\right)^2\left\{\frac{2+2x-2x^2}{(x^2+1)^2}\right\}$

(c) $\sin\left(\frac{2x-1}{x^2+1}\right)^2\left\{\frac{2+2x-x^2}{(x^2+1)}\right\}$

(d) none of these

One or More Than One Option(s) Correct Type

7. Let $y=\sqrt{x+\sqrt{x+\sqrt{x+\ldots\infty}}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{2y-1}$ (b) $\frac{x}{x+2y}$

(c) $\frac{1}{\sqrt{1+4x}}$ (d) $\frac{y}{2x+y}$

8. $\lim\limits_{x\to0}\left(\sum\limits_{r=1}^{n}r^{\operatorname{cosec}^2x}\right)^{\sin^2x}=$

(a) 0 (b) ∞ (c) n (d) $\frac{1}{n}$

9. If $f(x)=\frac{3x^2+ax+a+1}{x^2+x-2}$, then which of the following can be correct?

(a) $\lim\limits_{x\to1}f(x)$ exists $\Rightarrow a=-2$

(b) $\lim\limits_{x\to-2}f(x)$ exists $\Rightarrow a=13$

(c) $\lim\limits_{x\to1}f(x)=\frac{4}{3}$ (d) $\lim\limits_{x\to-2}f(x)=-\frac{1}{3}$

10. The value of a for which

$$\lim_{x\to 0}\frac{(e^x-1)^4}{\sin\left(\frac{x^2}{a^2}\right)\log_e\left\{1+\frac{x^2}{2}\right\}}=8, \text{ is}$$

(a) -2 (b) -1 (c) 1 (d) 2

11. If $\lim_{x\to 0}(\cos x + a\sin bx)^{1/x} = e^2$, then the values of a and b are

(a) $a=1, b=2$ (b) $a=2, b=\frac{1}{2}$

(c) $a=2\sqrt{2}, b=\frac{1}{\sqrt{2}}$ (d) $a=4, b=2$

12. If $m, n \in N$, $\lim_{x\to 0}\frac{\sin x^n}{(\sin x)^m}$ is equal to

(a) 1, if $n=m$ (b) 0, if $n>m$

(c) ∞, if $n<m$ (d) n/m, if $n<m$

13. Let $f(x)=\frac{x^2-9x+20}{x-[x]}$, (where $[\cdot]$ denotes the greatest integer function), then

(a) $\lim_{x\to 5^-} f(x)=0$ (b) $\lim_{x\to 5^+} f(x)=1$

(c) $\lim_{x\to 5} f(x)$ does not exist

(d) none of these

Comprehension Type

Let $a_1 > a_2 > a_3 > .... > a_n > 1$ and $p_1 > p_2 > p_3 > .... > p_n > 0$ such that $p_1 + p_2 + p_3 + ... + p_n = 1$. Also, $F(x) = (p_1a_1^x + p_2a_2^x + ... + p_na_n^x)^{1/x}$.

14. $\lim_{x\to 0^+} F(x)$ equals

(a) $p_1 \ln a_1 + p_2 \ln a_2 + ... + p_n \ln a_n$

(b) $a_1^{p_1} + a_2^{p_2} + ... + a_n^{p_n}$

(c) $a_1^{p_1}\cdot a_2^{p_2} .... a_n^{p_n}$ (d) $\sum_{r=1}^{n} a_r p_r$

15. $\lim_{x\to\infty} F(x)$ equals

(a) $\ln a_1$ (b) e^{a_n} (c) a_1 (d) a_n

Matrix Match Type

16. Match the columns:

Column-I		Column-II	
P.	If $f(x)=\left(\frac{\lvert x\rvert}{\lvert x\rvert+2}\right)^{-x}$, then	1.	$\lim_{x\to\infty} f(x)=e^2$
Q.	If $f(x)=\frac{(1+x)^{1/x}-e}{x}$, then	2.	$\lim_{x\to 0} f(x)=e^2$
R.	If $f(x)=\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}$, then	3.	$\lim_{x\to-\infty} f(x)=e^{-2}$
		4.	$\lim_{x\to 0} f(x)=-\frac{e}{2}$
		5.	$\lim_{x\to 0} f(x)<-1$

	P	Q	R
(a)	1, 3	4, 5	2
(b)	2, 3	4, 5	1
(c)	2, 3	1, 5	2
(d)	3, 4	1, 5	2

Numerical Value Type

17. The value of $\lim_{x\to 0} 27e^{-2}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x}$ must be

18. Let $\lim_{x\to 1}\frac{x^a-ax+a-1}{(x-1)^2}=f(a)$. Then the value of $f(4)$ is

19. If $L=\lim_{x\to 0}\frac{e^{-x^2/2}-\cos x}{x^3\sin x}$, then the value of $3L$ is

20. If $y=\sqrt{\left(\frac{1+\cos 2\theta}{1-\cos 2\theta}\right)}$, then $\frac{dy}{d\theta}$ at $\theta=\frac{3\pi}{4}$ is

◈◈

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SELF CHECK

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CONCEPT BOOSTER

Class XII

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR

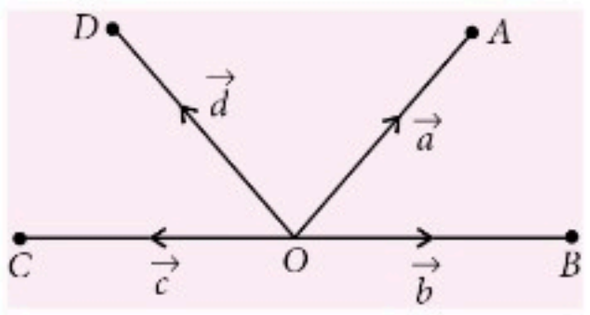
Vectors and Three - Dimensional Geometry

Vectors

A vector may be described as a quantity having both magnitude and direction. A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the initial point and B is called the terminal point. The magnitude of vector $\overrightarrow{AB}$ is expressed by $|\overrightarrow{AB}|$.

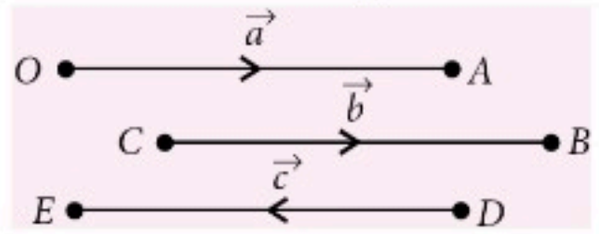
TYPES OF VECTORS

- **Position Vector:** If P is any point in the space, then the vector $\overrightarrow{OP}$ is called position vector of point P, where O is the origin of reference. Thus for any two points A and B in the space, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$.
- **Co-initial Vectors:** Vectors having same initial point are called co-initial vectors.

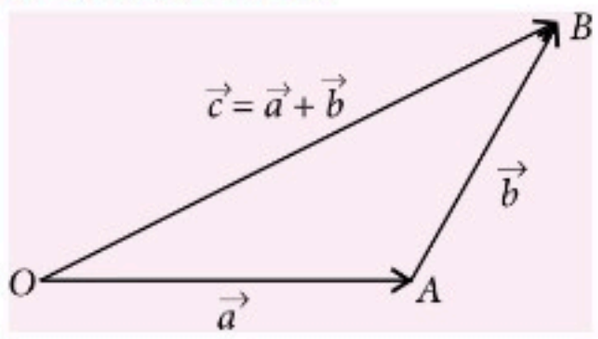


Here $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ and $\overrightarrow{OD}$ are co-initial vectors.

- **Equal Vectors :** Two vectors are said to be equal if and only if they have equal magnitudes and same direction.
- **Zero Vector (Null Vector):** A vector whose initial and terminal points are same, is called the null vector. Such vector has zero magnitude and no direction, and denoted by $\vec{0}$.
- **Like and Unlike Vectors:** Two vectors are said to be
  (i) Like, when they are in same direction.
  (ii) Unlike, when they are in opposite directions. $\vec{a}$ and $-\vec{a}$ are two unlike vectors as their directions are opposite, $\vec{a}$ and $3\vec{a}$ are like vectors.
- **Unit Vector :** A unit vector is a vector whose magnitude is unity. We write, unit vector in the direction of $\vec{a}$ as $\hat{a}$. Therefore $\hat{a}=\frac{\vec{a}}{|\vec{a}|}$.
- **Parallel Vectors:** Two or more vectors are said to be parallel, if they have the same support or parallel support. Parallel vectors may have equal or unequal magnitudes and direction may be same or opposite.



ADDITION OF TWO VECTORS



Let $\overrightarrow{OA}=\vec{a}, \overrightarrow{AB}=\vec{b}$ and $\overrightarrow{OB}=\vec{c}$.

Here $\vec{c}$ is sum (or resultant) of vectors $\vec{a}$ and $\vec{b}$. It is to be noticed that the initial point of $\vec{b}$ coincides with the terminal point of $\vec{a}$ and the line joining the initial point of $\vec{a}$ to the terminal point of $\vec{b}$ represents vector $\vec{a}+\vec{b}$ in magnitude and direction.

*Alok Kumar, a B.Tech from IIT Kanpur and INMO 4th ranker of his time, has been training IIT and Olympiad aspirants for close to two decades now. His students have bagged AIR 1 in IIT JEE and also won medals for the country at IMO. He has also taught at Maths Olympiad programme at Cornell University, USA and UT, Dallas. He has been regularly proposing problems in international Mathematics journals.

PROPERTIES

- $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (Vector addition is commutative)
- $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$ (Vector addition is associative)
- $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$, equality holds when $\vec{a}$ and $\vec{b}$ are like vectors.
- $|\vec{a}+\vec{b}|\ge||\vec{a}|-|\vec{b}||$, equality holds when $\vec{a}$ and $\vec{b}$ are unlike vectors.
- $\vec{a}+0=\vec{a}=0+\vec{a}$
- $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$

MULTIPLICATION OF VECTOR BY SCALARS

If $\vec{a}$ is a vector and m is a scalar, then $m\vec{a}$ is a vector parallel to $\vec{a}$ whose magnitude is $|m|$ times that of $\vec{a}$. This multiplication is called scalar multiplication. If $\vec{a}$ and $\vec{b}$ are vectors and m, n are scalars, then

- $m(\vec{a})=(\vec{a})m=m\vec{a}$
- $m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$
- $(m+n)\vec{a}=m\vec{a}+n\vec{a}$
- $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$

SECTION FORMULA

If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the position vector of a point which divides AB in the ratio $m:n$ is given by $\vec{r}=\dfrac{n\vec{a}+m\vec{b}}{m+n}$.

Note : Position vector of mid-point of $AB=\dfrac{\vec{a}+\vec{b}}{2}$.

SCALAR PRODUCT OF TWO VECTORS (DOT PRODUCT)

The scalar product, $\vec{a}\cdot\vec{b}$ of two non-zero vectors $\vec{a}$ and $\vec{b}$ is defined as $|\vec{a}||\vec{b}|\cos\theta$, $0\le\theta\le\pi$ where θ is the angle between the two vectors, when drawn with same initial point.

If at least one of $\vec{a}$ and $\vec{b}$ is a zero vector, then $\vec{a}\cdot\vec{b}$ is defined as zero.

PROPERTIES

- $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ (Scalar product is commutative)
- $\vec{a}^2=\vec{a}\cdot\vec{a}=|\vec{a}|^2$
- $(m\vec{a})\cdot\vec{b}=m(\vec{a}\cdot\vec{b})=\vec{a}\cdot(m\vec{b})$, where m is a scalar.
- $\theta=\cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$
- $\vec{a}\cdot\vec{b}=0\Leftrightarrow$Vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other. [Provided $\vec{a},\vec{b}$ are non-zero vectors]
- $\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$
- $\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0$
- $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
- $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=|\vec{a}|^2-|\vec{b}|^2$
- Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$. Then, $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$
- Projection of $\vec{a}$ on $\vec{b}=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

VECTOR PRODUCT OF TWO VECTORS (CROSS PRODUCT)

If $\vec{a}$ and $\vec{b}$ are two vectors and θ is the angle between them, then $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ forms a right handed screw system .

PROPERTIES

- $\vec{a}\times\vec{b}=0\Leftrightarrow\vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a}\ne0,\ \vec{b}\ne0)$ *i.e.*, $\vec{a}=K\vec{b}$, where K is a scalar.
- $\vec{a}\times\vec{b}\ne\vec{b}\times\vec{a}$ (Not commutative)
- $(m\vec{a})\times\vec{b}=\vec{a}\times(m\vec{b})=m(\vec{a}\times\vec{b})$, where m is a scalar.
- $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$ (Distributive)
- $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$
- $\hat{i}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$, $\hat{k}\times\hat{i}=\hat{j}$
- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$.
- Geometrically $|\vec{a}\times\vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.
- Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ and $\vec{d}_2$ is given by $\dfrac{1}{2}|\vec{d}_1\times\vec{d}_2|$.
- Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\hat{n}=\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.
- A vector of magnitude r and perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$.
- If 0 is the angle between $\vec{a}$ and b , then $\sin\theta=\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$.
- **Lagrange's identity :** For any two vectors $\vec{a}$ and $\vec{b}$; $(\vec{a}\times\vec{b})^2=|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2=\begin{vmatrix}\vec{a}\cdot\vec{a}&\vec{a}\cdot\vec{b}\\\vec{a}\cdot\vec{b}&\vec{b}\cdot\vec{b}\end{vmatrix}$

MAGAZINE KING

- **Formulation of vector product in terms of scalar product**

  The vector product $\vec{a}\times\vec{b}$ is the vector $\vec{c}$, such that

  (i) $|\vec{c}|=\sqrt{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2}$

  (ii) $\vec{c}\cdot\vec{a}=0, \vec{c}\cdot\vec{b}=0$

  (iii) $\vec{a}, \vec{b}, \vec{c}$ form a right handed system

AREA OF TRIANGLE

If $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vectors of three points A, B and C, then the area of triangle

$$ABC=\frac{1}{2}\left[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right].$$

The points A, B and C are collinear if

$$\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0$$

SCALAR TRIPLE PRODUCT

The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as $(\vec{a}\times\vec{b})\cdot\vec{c}=|\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$

where, θ is the angle between $\vec{a}$ and $\vec{b}$ and ϕ is the angle between $(\vec{a}\times\vec{b})$ and $\vec{c}$. It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as box product.

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$; $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then $[\vec{a}\ \vec{b}\ \vec{c}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

PROPERTIES

- In a scalar triple product, the position of dot and cross can be interchanged.

  i.e., $\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$

  or $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]$

- $\vec{a}\cdot(\vec{b}\times\vec{c})=-\vec{a}\cdot(\vec{c}\times\vec{b})$
- Scalar triple product geometrically represents the volume of the parallelepiped whose three coterminous edges are represented by $\vec{a}$, $\vec{b}$ and $\vec{c}$ *i.e.*, $V=[\vec{a}\ \vec{b}\ \vec{c}]$.
- Vectors $\vec{a}, b, \vec{c}$ are coplanar $\Leftrightarrow$ $[\vec{a}\ b\ \vec{c}]=0$.
- Scalar product of three vectors, two of which are equal or parallel is 0.
- If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then $[\vec{a}\ \vec{b}\ \vec{c}]>0$ for right handed system and $[\vec{a}\ \vec{b}\ \vec{c}]<0$ for left handed system.
- $[\hat{i}\ \hat{j}\ \hat{k}]=1$
- $[K\vec{a}\ \vec{b}\ \vec{c}]=K[\vec{a}\ \vec{b}\ \vec{c}]$
- $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}]=[\vec{a}\ \vec{c}\ \vec{d}]+[\vec{b}\ \vec{c}\ \vec{d}]$
- The volume of the tetrahedron $OABC$ with O as origin and the position vectors of A, B and C being $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively is given by $V=\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$.
- The position vector of the centroid of a tetrahedron if the position vectors of its angular vertices are $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are given by $\frac{1}{4}[\vec{a}+\vec{b}+\vec{c}+\vec{d}]$.

VECTOR TRIPLE PRODUCT

The vector triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as $\vec{a}\times(\vec{b}\times\vec{c})$. If at least one $\vec{a}, \vec{b}$ and $\vec{c}$ is a zero vector or $\vec{b}$ and $\vec{c}$ are collinear vectors or $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$, only then $\vec{a}\times(\vec{b}\times\vec{c})=\vec{0}$. In all other cases $\vec{a}\times(\vec{b}\times\vec{c})$ will be a non-zero vector in the plane of non-collinear vectors and perpendicular to the vector $\vec{a}$.

Thus we can take $\vec{a}\times(\vec{b}\times\vec{c})=\lambda\vec{b}+\mu\vec{c}$, for some scalars λ and μ.

Since, $\vec{a}\perp\vec{a}\times(\vec{b}\times\vec{c})\Rightarrow a\cdot(\vec{a}\times(\vec{b}\times\vec{c}))=0$

$\Rightarrow \lambda(\vec{a}\cdot\vec{b})+\mu(\vec{a}\cdot\vec{c})=0$

$\Rightarrow \lambda=(\vec{a}\cdot\vec{c})\alpha, \mu=-(\vec{a}\cdot\vec{b})\alpha$, for some scalar α.

Hence, $\vec{a}\times(\vec{b}\times\vec{c})=\alpha[(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c})]$, for any vectors $\vec{a}, \vec{b}$ and $\vec{c}$ satisfying the conditions given in the beginning. In particular if we take, $\vec{a}=\vec{b}=\hat{i}, \vec{c}=\hat{j}$, then $\alpha=1$.

Hence, $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

SCALAR AND VECTOR PRODUCT OF FOUR VECTORS

- If $\vec{p}, \vec{q}, \vec{r}, \vec{s}$ are four vectors then the scalar product is computed as $(\vec{p}\times\vec{q})\cdot(\vec{r}\times\vec{s})$ and defined by

  $$(\vec{p}\times\vec{q})\cdot(\vec{r}\times\vec{s})=\begin{vmatrix} \vec{p}\cdot\vec{r} & \vec{p}\cdot\vec{s} \\ \vec{q}\cdot\vec{r} & \vec{q}\cdot\vec{s} \end{vmatrix}$$

- Vector product of four vectors are computed as

  $(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{s})=[\vec{p}\ \vec{q}\ \vec{s}]\vec{r}-[\vec{p}\ \vec{q}\ \vec{r}]\vec{s}$

  $=[\vec{p}\ \vec{r}\ \vec{s}]\vec{q}-[\vec{q}\ \vec{r}\ \vec{s}]\vec{p}$

LINEAR COMBINATIONS

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}, \ldots$, then the vector $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}+\ldots$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}, \ldots$ for any $x, y, z, \ldots \in R$.

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- **Fundamental theorem in plane :** Let $\vec{a}, \vec{b}$ be non-zero, non-collinear vectors. Then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ *i.e.*, there exist some unique $x, y \in R$ such that $x\vec{a} + y\vec{b} = \vec{r}$.
- **Fundamental theorem in space :** Let $\vec{a}, \vec{b}, \vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$ can be uniquely expressed as a linear combination of $\vec{a}$, $\vec{b}$, $\vec{c}$ *i.e.,* there exist some unique $x, y, z \in R$ such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.
- **Linearly independent and dependent vectors**

(i) If $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are n non-zero vectors, and $k_1, k_2, ..., k_n$ are n scalars and if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + ... + k_n\vec{x}_n = 0$
$\Rightarrow \quad k_1 = 0, k_2 = 0, ..., k_n = 0$
then we say that vectors $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are linearly independent vectors.

(ii) If $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are not linearly independent, then they are said to be linearly dependent vectors. *i.e.*, if $k_1\vec{x}_1 + k_2\vec{x}_2 + ... + k_n\vec{x}_n = 0$ and if there exists at least one $k_r \neq 0$, then $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are said to be linearly dependent vectors.

- If $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$, then $\vec{a}$ is expressed as a linear combination of vectors $\hat{i}, \hat{j}, \hat{k}$. Also, $\vec{a}, \hat{i}, \hat{j}, \hat{k}$ form a linearly dependent set of vectors. In general, every set of four vectors forms a linearly dependent system.
- $\hat{i}, \hat{j}, \hat{k}$ are linearly independent set of vectors.
  For $K_1\hat{i} + K_2\hat{j} + K_3\hat{k} = 0 \Rightarrow K_1 = 0 = K_2 = K_3$.
- Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent *i.e.*, $\vec{a}$ is parallel to $\vec{b}$ *i.e.*, $\vec{a} \times \vec{b} = 0$. Conversely, if $\vec{a} \times \vec{b} \neq 0$, then $\vec{a}$ and $\vec{b}$ are linearly independent.

COLLINEARITY AND COPLANARITY OF POINTS

The necessary and sufficient condition for three points with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ to be collinear is that there exist scalars x, y, z (not all zero) such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where $x + y + z = 0$

The necessary and sufficient condition for four points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ to be coplanar is that there exist scalars x, y, z, u (not all zero) such that $x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}$ where $x + y + z + u = 0$.

RECIPROCAL SYSTEM OF VECTORS

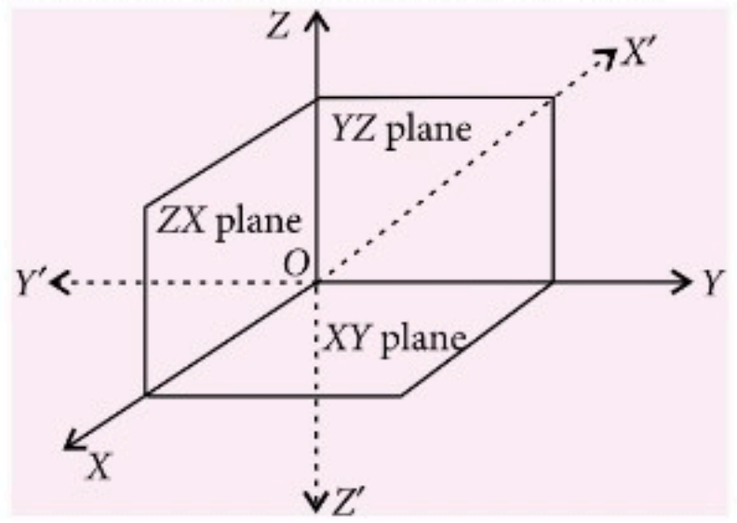
- $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non-coplanar vectors such that $\vec{a} \cdot \vec{a}' = \vec{b} \cdot \vec{b}' = \vec{c} \cdot \vec{c}' = 1$, then the two systems are called reciprocal system of vectors.

Note : $\vec{a}' = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{b}' = \dfrac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{c}' = \dfrac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

Three - Dimensional Geometry

RECTANGULAR COORDINATE SYSTEM IN SPACE

Let 'O' be any point in space and three lines are perpendicular to each other. These lines are known as coordinate axes and O is called origin. The planes XY, YZ, ZX are known as the coordinate planes.



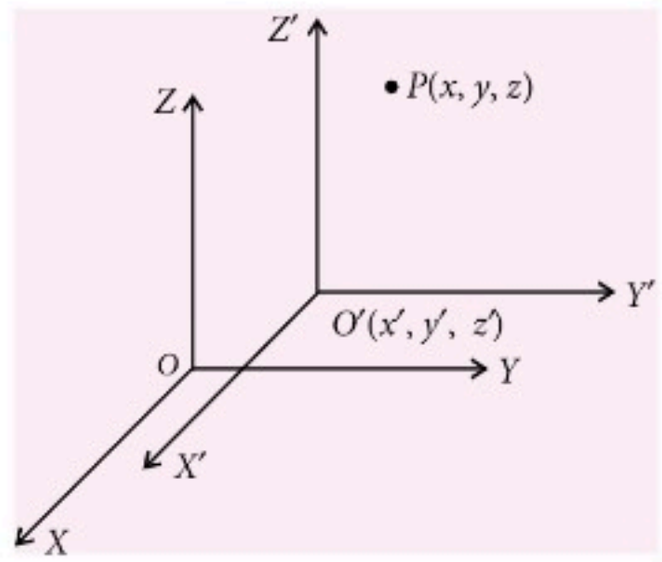
COORDINATES OF A POINT IN SPACE

Consider a point P in space whose position is given by triad (x, y, z) where x, y, z are perpendicular distance from YZ-plane, ZX-plane and XY-plane respectively.

If we assume $\hat{i}, \hat{j}, \hat{k}$ unit vectors along OX, OY, OZ respectively, then position vector of point P is $x\hat{i} + y\hat{j} + z\hat{k}$ or simply (x, y, z).

SHIFTING OF ORIGIN

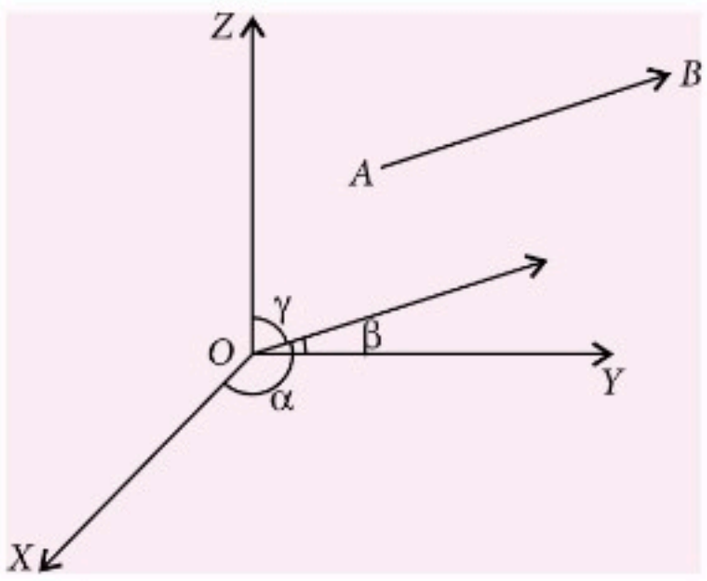
Shifting the origin to another point without changing the directions of the axes is called the translation of axes. Let the origin O be shifted to another point without changing the direction of axes.



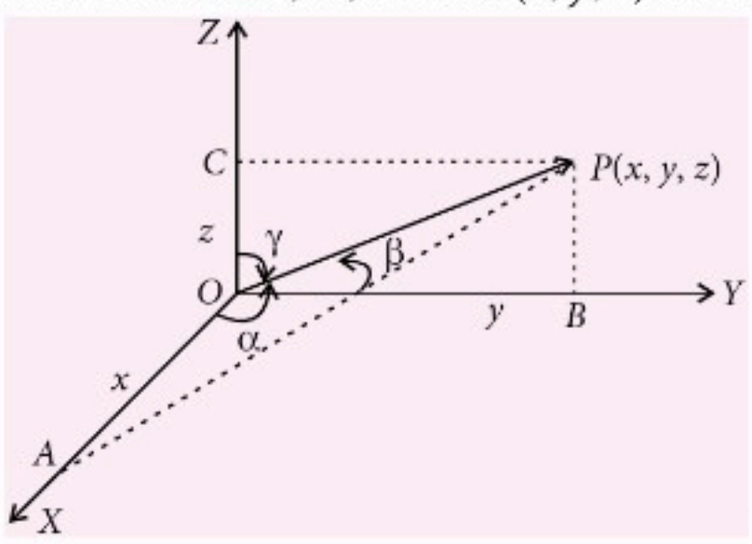
Let the new coordinate frame be $O'X'Y'Z'$. Let $P(x, y, z)$ be a point with respect to the coordinate frame $OXYZ$. Then, coordinate of point P w.r.t. new coordinate frame $O'X'Y'Z'$ is (x_1, y_1, z_1), where, $x_1 = x - x'$, $y_1 = y - y'$, $z_1 = z - z'$ where O is shifted at $O'(x', y', z')$.

DIRECTION COSINES & DIRECTION RATIOS OF A LINE

If α, β, γ be the angles which a given directed line makes with the positive direction of the x, y, z coordinate axes respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the given line and are generally denoted by l, m, n respectively. Thus $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$. By definition it follows that the direction cosine of the x-axis is cos 0°, cos 90°, cos 90° *i.e.*, (1, 0, 0). Similarly direction cosines of the y-axis and z-axis are respectively (0, 1, 0) and (0, 0, 1).



- The unit vector along the line can be written as $l\hat{i}+m\hat{j}+n\hat{k}$.
- If d.c.'s of line AB is (l, m, n), then d.c.'s of line BA will be $(-l, -m, -n)$.
- Let OP be any line through the origin O which has direction cosines l, m, n. Let $P(x, y, z)$ and $OP = r$.



Then $OP^2 = x^2 + y^2 + z^2 = r^2$ ...(1)

From P draw PA, PB, PC perpendicular on the coordinate axes, so that $OA = x$, $OB = y$, $OC = z$

Also, $\angle POA = \alpha$, $\angle POB = \beta$ and $\angle POC = \gamma$.

From ΔAOP, $l = \cos\alpha = \frac{x}{r} \Rightarrow x = lr$

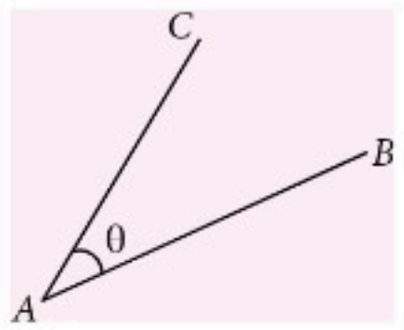
Similarly, $y = mr$ and $z = nr$

Hence from (1),

$r^2(l^2 + m^2 + n^2) = r^2 \quad \Rightarrow \quad l^2 + m^2 + n^2 = 1$

- If the coordinates of any point P be (x, y, z) and l, m, n be the direction cosines of the line OP, O being the origin, then (lr, mr, nr) will give us the coordinates of a point on the line OP which is at a distance r from (0, 0, 0).

ANGLE BETWEEN TWO LINES



Let θ be the angle between two straight lines AB and AC whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively, is given by

$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$.

If direction ratios of two lines are a_1, b_1, c_1 and a_2, b_2, c_2 are given, then angle between two lines is given by $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\cdot\sqrt{a_2^2 + b_2^2 + c_2^2}}$

Also, we have,

$$\sin^2\theta = 1 - \cos^2\theta = (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$$

$$= (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2$$

$$\Rightarrow \sin\theta = \pm\sqrt{\sum(l_1m_2 - l_2m_1)^2}$$

Note : (i) If the given lines are perpendicular, then $\theta = 90°$ *i.e.*, $\cos\theta = 0$

$\Rightarrow \quad l_1l_2 + m_1m_2 + n_1n_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) If the given lines are parallel, then $\theta = 0°$ *i.e.*, $\sin\theta = 0$

$$\Rightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} \text{ or } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

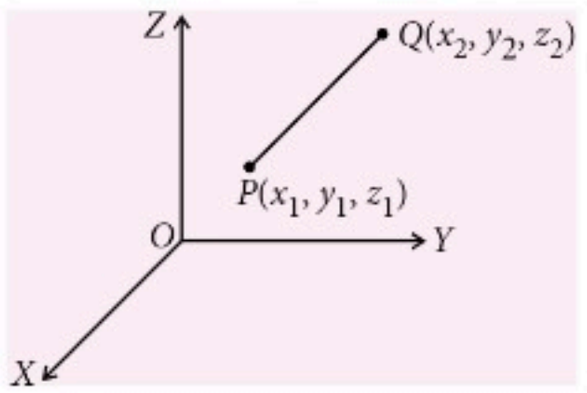
DIRECTION COSINES OF A LINE PASSING THROUGH TWO POINTS

The direction ratios of line PQ joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1 = a$(say), $y_2 - y_1 = b$(say) and $z_2 - z_1 = c$(say). Then direction cosines are

$$l = \pm\frac{(x_2 - x_1)}{\sqrt{\sum(x_2 - x_1)^2}},$$

$$m = \pm\frac{(y_2 - y_1)}{\sqrt{\sum(y_2 - y_1)^2}},$$

$$n = \pm\frac{(z_2 - z_1)}{\sqrt{\sum(z_2 - z_1)^2}}$$



THE STRAIGHT LINE

Straight line in three dimensional geometry is defined as intersection of two planes. So general equation of straight line is stated as the equations of both planes together *i.e.*, general equation of straight line is

$a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ ....(1)

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So, equation (1) represents straight line which is obtained by intersection of two planes.

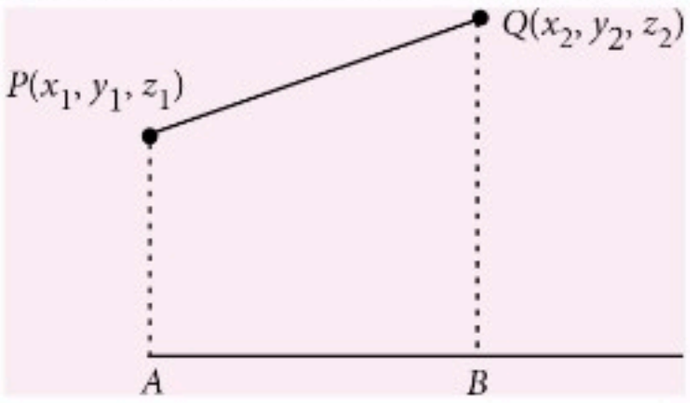
EQUATION OF STRAIGHT LINE IN DIFFERENT FORMS

- Equation of straight line passing through point $P(x_1, y_1, z_1)$ and whose direction cosines are l, m, n is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r$.
- Vector equation of straight line is $\vec{r}=\vec{r}_1+\lambda\vec{b}$ where $\vec{r}_1$ is the position vector of a point in the straight line and $\vec{b}$ is a vector parallel to the straight line.
- Equation of straight line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$.
- Equation of straight line passing through two points $P(\vec{r}_1)$ and $Q(\vec{r}_2)$ is $\vec{r}=\vec{r}_1+\lambda(\vec{r}_2-\vec{r}_1)$.

  Note : The general coordinates of a point on a line is given by $(x_1 + lr, y_1 + mr, z_1 + nr)$ where r is distance between the point (x_1, y_1, z_1) and point whose coordinates is to be written.

PROJECTION OF A LINE SEGMENT ON A GIVEN LINE

Projection of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose direction cosines are l, m, n is $AB = |l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$. Here PQ and AB need not be coplanar.



COPLANARITY OF TWO LINES

Cartesian form : If the lines are

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

then condition of coplanarity is

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

and the equation of plane containing them is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0 \text{ or } \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

Vector form : If the lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ are coplanar, then $\left[\vec{a}_1\ \vec{b}_1\ \vec{b}_2\right]=\left[\vec{a}_2\ \vec{b}_1\ \vec{b}_2\right]$ and the equation of plane containing them is

$\left[\vec{r}\ \vec{b}_1\ \vec{b}_2\right]=\left[\vec{a}_1\ \vec{b}_1\ \vec{b}_2\right]$ or $\left[\vec{r}\ \vec{b}_1\ \vec{b}_2\right]=\left[\vec{a}_2\ \vec{b}_1\ \vec{b}_2\right]$.

SHORTEST DISTANCE BETWEEN TWO SKEW LINES (NON-PARALLEL, NON-INTERSECTING LINES)

Two non-parallel, non-intersecting lines are called skew lines. The shortest distance between these two lines is the distance of the intercepting portion of a line perpendicular to both the lines.

Let the equation of two non-intersecting lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}=r_1(\text{say}) \quad ...(1)$$

$$\text{and } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}=r_2(\text{say}) \quad ...(2)$$

Any point on line (1) is $P(x_1 + l_1r_1, y_1 + m_1r_1, z_1 + n_1r_1)$ and on line (2) is $Q(x_2 + l_2r_2, y_2 + m_2r_2, z_2 + n_2r_2)$.

Let PQ be the line of shortest distance, its direction ratios will be

$[(l_1r_1 + x_1 - x_2 - l_2r_2), (m_1r_1 + y_1 - y_2 - m_2r_2), (n_1r_1 + z_1 - z_2 - n_2r_2)]$

This line is perpendicular to both the given lines. By using condition of perpendicularity we obtain two equations in r_1 and r_2. So by solving these, values of r_1 and r_2 and subsequently point P and Q can be found. The distance PQ is the shortest distance.

Cartesian form : The shortest distance between two lines is given by

$$\frac{1}{\sqrt{\sum(l_1m_2-l_2m_1)^2}}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}.$$

Vector form : The shortest distance between two lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ is $\left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$.

If any straight line is given in general form, then it can be transformed into symmetrical form and we can further proceed.

SHORTEST DISTANCE BETWEEN TWO PARALLEL LINES

If two lines l_1 and l_2 are parallel, then they are coplanar.

Let the lines be given by $\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $\vec{r}=\vec{a}_2+\mu\vec{b}$, where $\vec{a}_1$ is the position vector of a point on l_1 and $\vec{a}_2$ is the position vector of a point on l_2.

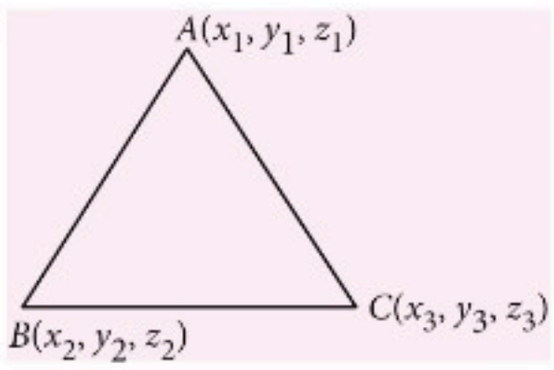
The shortest distance between the given parallel lines

is $d = \left|\dfrac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$.

AREA OF A TRIANGLE

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix},$$

$$\Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$



So, area of ΔABC is given by the relation

$$\Delta^2 = \Delta_x{}^2 + \Delta_y{}^2 + \Delta_z{}^2$$

THE PLANE

Consider the locus of a point $P(x, y, z)$. If x, y, z are allowed to vary with any restriction for their different combinations, we have a set of points for P. The surface on which these points lie, is called the locus of P. It may be a plane or any curved surface. If Q be any other point on its locus and all points of the straight line PQ lie on it, then it is a plane. In other words, for the plane the straight line PQ, however small and in whatever direction will completely lie on the locus, otherwise it will be a curved surface.

EQUATION OF A PLANE IN NORMAL FORM, PASSING THROUGH A FIXED POINT

Cartesian form : General equation of a plane is $ax + by + cz + d = 0$. (where a, b, c gives the direction ratios of the normal to the plane).

Equation of the plane in normal form is $lx + my + nz = p$ where p is the length of the normal from the origin to the plane and (l, m, n) be the direction cosines of the normal .

Vector form : Equation of the plane in vector form, $\vec{r} \cdot \hat{n} = d$, where $\hat{n}$ is a unit vector perpendicular to the plane and d is length of perpendicular drawn from origin to the plane.

Equation of YZ plane is $x = 0$,

equation of plane parallel to YZ plane is $x = d$.

Equation of ZX plane is $y = 0$,

equation of plane parallel to ZX plane is $y = d$.

Equation of XY plane is $z = 0$,

equation of plane parallel to XY plane is $z = d$.

Four points namely $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ will be coplanar if one point lies on the plane passing through other three points.

EQUATION OF A PLANE PERPENDICULAR TO A GIVEN LINE AND PASSING THROUGH A GIVEN POINT

Cartesian form : The equation to the plane passing through $P(x_1, y_1, z_1)$ and is perpendicular to given line having direction ratios (a, b, c) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

Vector form : The equation to the plane passing through a point whose position vector is $\vec{r}_1$ and unit vector perpendicular to the plane is $\hat{n}$, is $(\vec{r} - \vec{r}_1) \cdot \hat{n} = 0$.

EQUATION OF A PLANE PASSING THROUGH THREE NON COLLINEAR POINTS

The equation of the plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3)

is $\begin{vmatrix} (x - x_1) & (y - y_1) & (z - z_1) \\ (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \end{vmatrix} = 0.$

INTERCEPT FORM OF THE EQUATION OF A PLANE

Intercept form of the equation of a plane is given by $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ where a, b, c are x-intercept, y-intercept and z-intercept respectively.

ANGLE BETWEEN TWO PLANES

Cartesian form : Angle between the planes is defined as angle between normals to the planes drawn from any point.

Angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$$\cos^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\left(a^2{}_1 + b^2{}_1 + c^2{}_1\right)\left(a^2{}_2 + b^2{}_2 + c^2{}_2\right)}}\right).$$

Vector form : Angle between the planes is

$$\cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right).$$

where $\vec{n}_1$ and $\vec{n}_2$ are vectors perpendicular to the planes.

Note : (i) If $a_1a_2 + b_1b_2 + c_1c_2 = 0$, then the planes are perpendicular to each other.

(ii) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, then the planes are parallel to each other.

DISTANCE OF A POINT FROM A PLANE

Cartesian form : The length of the perpendicular of the point $P(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$ is $\dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

Vector form : The length of the perpendicular from the point $P(\vec{r}_1)$ to the plane $\vec{r}.\hat{n}=d$ is $|\vec{r}_1.\hat{n}-d|$.

DISTANCE BETWEEN TWO PARALLEL PLANES

The distance between two parallel planes is the algebraic difference of perpendicular distances on the planes from origin.

Let two parallel planes be $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$.

Then distance between them is given by $\frac{|d_2 - d_1|}{\sqrt{a^2+b^2+c^2}}$.

POSITION OF TWO POINTS W.R.T. A PLANE

Two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on the same or opposite sides of a plane $ax + by + cz + d = 0$ according to $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of same or opposite signs. The plane divides the line joining the points P and Q externally or internally according to P and Q are lying on same or opposite sides of the plane.

EQUATION OF THE PLANE THROUGH A GIVEN LINE

- If equation of line is given in general form as $a_1x+b_1y+c_1z+d_1=0, a_2x+b_2y+c_2z+d_2=0$, then the equation of plane passing through this line is $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$.
- If equation of the line is given in symmetrical form as $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, then equation of plane is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b, c are given by $al + bm + cn = 0$.

PLANE PASSING THROUGH THE INTERSECTION OF TWO GIVEN PLANES

Equation of plane passing through the line of intersection of two planes $u = 0$ and $v = 0$ is $u + \lambda v = 0$.

ANGLE BETWEEN A LINE AND A PLANE

If equation of a plane is $ax + by + cz + d = 0$, then direction ratios of normal to this plane are a, b, c.

The equation of straight line is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, then angle between normal to the plane and straight line is given by $\sin\theta=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$, where θ is the angle between the plane and the line.

The plane and the straight line will be parallel if $al + bm + cn = 0$.

The plane and the straight line will be perpendicular if $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$.

EQUATION OF THE PLANES BISECTING THE ANGLES BETWEEN TWO GIVEN PLANES

- **Cartesian form :** Equation of two planes bisecting the angles between the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by
$$\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}=\pm\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$$
- **Vector form :** Let the two planes be $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$, then equation of the planes bisecting the angles of given planes is given by
$$\frac{|\vec{r}\cdot\vec{n}_1-d_1|}{|\vec{n}_1|}=\frac{|\vec{r}\cdot\vec{n}_2-d_2|}{|\vec{n}_2|} \text{ or } \vec{r}\cdot(\vec{n}_1\pm\vec{n}_2)=\frac{d_1}{|\vec{n}_1|}\pm\frac{d_2}{|\vec{n}_2|}$$

PROBLEMS

Single Correct Answer Type

1. Let $a, b > 0$ and $\vec{\alpha}=\frac{\hat{i}}{a}+\frac{4\hat{j}}{b}+b\hat{k}$ & $\vec{\beta}=b\hat{i}+a\hat{j}+\frac{1}{b}\hat{k}$, then the maximum value of $\frac{10}{5+\vec{\alpha}\cdot\vec{\beta}}$ is

(a) 1 (b) 2 (c) 4 (d) 8

2. Equation of a plane bisecting the angle between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$ is

(a) $5x - y - 4z - 45 = 0$

(b) $5x - y - 4z - 3 = 0$

(c) $23x + 13y + 32z - 45 = 0$

(d) $23x - 13y + 32z + 5 = 0$

3. If the angle θ between the line $\frac{x+1}{1}=\frac{y-1}{2}=\frac{z-2}{2}$ and the plane $2x-y+\sqrt{\lambda}z+4=0$ is such that $\sin\theta = 1/3$, then the value of λ is

(a) 3/4 (b) −4/3 (c) 5/3 (d) −3/5

4. Two adjacent sides of a parallelogram $ABCD$ are given by $\overrightarrow{AB}=2\hat{i}+10\hat{j}+11\hat{k}$ and $\overrightarrow{AD}=-\hat{i}+2\hat{j}+2\hat{k}$. The side AD is rotated by an acute angle α in the plane of parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of angle α is given by

(a) $\frac{8}{9}$ (b) $\frac{\sqrt{17}}{9}$ (c) $\frac{1}{9}$ (d) $\frac{4\sqrt{5}}{9}$

5. Equation of the plane through (3, 4, −1) which is parallel to the plane $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+7=0$ is

(a) $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+11=0$

CONCEPT MAP

STATISTICS

Class XI | Class

Measures of Central Tendency

Mean, Median, Mode

Analysis of Frequency Distributions

Coefficient of variation (C.V.) $= \frac{\text{S.D.}}{\text{Mean}} \times 100 = \frac{\sigma}{\bar{x}} \times 100,\ \bar{x} \neq 0$

- The data having higher C.V. will be more variable and the data having lower C.V. will be more consistent.

For Ungrouped Data

Let $x_1, x_2, \ldots\ldots, x_n$ be n observations, then mean deviation

- About mean $= \frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}|$, where $\bar{x}$ = Mean
- About median $= \frac{1}{n}\sum_{i=1}^{n}|x_i - M|$, where M = Median

For Grouped Data

Let $x_1, x_2, \ldots, x_n$ be a set of n observations occurring with frequencies $f_1, f_2, \ldots, f_n$ respectively, then mean deviation

- About mean, $\bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_i|x_i - \bar{x}|$;
- About median, $M = \frac{1}{N}\sum_{i=1}^{n} f_i\,|x_i - M|$

Here, x_i are the mid-points of classes and $N = \sum_{i=1}^{n} f_i$ = Sum of frequencies.

Short Cut Method

- Mean deviation about mean, $\bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_i|x_i - \bar{x}|$, where mean, $(\bar{x}) = a + \frac{\sum_{i=1}^{n} f_i u_i}{N} \times h$, where $u_i = \frac{x_i - a}{h}$, a is assumed mean and h is width of class-intervals.
- Mean deviation about median, $M = \frac{1}{N}\sum_{i=1}^{n} f_i\,|x_i - M|$, where median, $(M) = l + \frac{\frac{N}{2} - cf}{f} \times h$, where l, f and cf are lower limit, frequency and cumulative frequency.

Measures of Dispersion

Range

Difference between two extreme observations of the given distribution.
Range = Maximum Value – Minimum Value

Quartiles

Quartiles are those values of the variate which divide the distribution into four equal parts.

Mean Deviation

It is the arithmetic mean of the absolute values of deviations about some point (mean or median or mode).

Mean Deviation $= \frac{\text{Sum of Deviations}}{\text{Number of Observations}}$

Standard Deviation

Positive square root of the variance is known as standard deviation and it is denoted by σ.

For ungrouped data	$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$
For grouped data	$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}$
Shortcut method	$\sigma = \frac{h}{N}\left[\sqrt{N\sum_{i=1}^{n} f_i u_i^2 - \left(\sum_{i=1}^{n} f_i u_i\right)^2}\right]$ where $u_i = \frac{x_i - a}{h}$, a = assumed mean, h = width of class-intervals

Variance of Combined Series

Let $\bar{x}_1$ and $\bar{x}_2$ be means of two groups with standard deviations σ_1, σ_2 and number of elements n_1 and n_2, then variance is given by

$$(\sigma^2) = (\text{S.D.})^2 = \frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}$$

where $d_1 = \bar{x}_1 - \bar{x}$, $d_2 = \bar{x}_2 - \bar{x}$ and $\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

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PLANE

CONCEPT MAP

ss XI | Class XII

DEFINITION

A plane is a surface such that line joining any two points totally lies in it.

EQUATION OF PLANE IN DIFFERENT FORMS

Normal Form

- **Vector form :** $\vec{r} \cdot \hat{n} = d$, where d is distance of plane from origin and $\hat{n}$ is unit vector normal to the plane.
- **Cartesian form :** $lx + my + nz = d$, where l, m, n are direction cosines of normal to plane.

One Point Form

- **Vector form :** $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$
- **Cartesian form :** $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b, c are dr's of normal to plane.

Equation of Plane Through Three non-collinear Points

- **Vector form :** $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$, where $\vec{a}, \vec{b}$ and $\vec{c}$ are position vectors of given points.
- **Cartesian form :** If (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), be three points then $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$

Intercept Form

If intercepts on x, y, z axes are a, b, c, then $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Equation of Plane Passing Through Intersection of Two Planes

- **Vector form :** Equation of plane passing through the intersection of the plane $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is given by $(\vec{r} \cdot \vec{n}_1 - d_1) + \lambda(\vec{r} \cdot \vec{n}_2 - d_2) = 0$ *i.e.*, $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$
- **Cartesian form :** The equation of plane passing through the intersection of $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by $(a_1x + b_1y + c_1z + d_1) + \mu(a_2x + b_2y + c_2z + d_2) = 0$

Angle Between Two Planes

Vector form : For $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ then

$\cos\theta = \pm \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

If $\vec{n}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{n}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

- Planes are perpendicular if
  $a_1a_2 + b_1b_2 + c_1c_2 = 0$ (Cartesian form)
  and $\vec{n}_1 \cdot \vec{n}_2 = 0$ (Vector form)
- Planes are parallel if
  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (Cartesian form)
  and $\vec{n}_1 = \lambda \vec{n}_2$ (Vector form)

Distance of a Point from a Plane

Vector form : $\frac{|\vec{a} \cdot \vec{n} - d|}{|\vec{n}|}$

where $\vec{a}$ = position vector of the point and plane is $\vec{r} \cdot \vec{n} = d$

Cartesian form : Perpendicular distance of a point (x_1, y_1, z_1) from a plane $ax + by + cz + d = 0$ is

$p = \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$

Distance Between Parallel Planes

If $\vec{r} \cdot \vec{n} = d_1$ and $\vec{r} \cdot \vec{n} = d_2$ be two parallel planes then

$d = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$ where $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$

Equation of Bisector of Two Planes

Vector form : $r \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ then,

$\frac{|\vec{r} \cdot \vec{n}_1 - d_1|}{|\vec{n}_1|} = \frac{|\vec{r} \cdot \vec{n}_2 - d_2|}{|\vec{n}_2|}$

Cartesian form : If $a_1x + b_1y + c_1y + d_1 = 0$ and $a_2x + b_2y + c_2y + d_2 = 0$ then

$\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} = \pm \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$

Image of a Point in a Plane

$\frac{x' - x_1}{a} = \frac{y' - y_1}{b} = \frac{z' - z_1}{c} = \frac{-2|ax_1 + by_1 + cz_1 + d|}{a^2 + b^2 + c^2}$

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(b) $\vec{r}\cdot(3\hat{i}+4\hat{j}-\hat{k})+11=0$

(c) $\vec{r}\cdot(3\hat{i}+4\hat{j}-\hat{k})+7=0$

(d) $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})-7=0$

6. For unit vectors $\vec{b}$ and $\vec{c}$ and for any non zero vector $\vec{a}$, the value of $\{[(\vec{a}+\vec{b})+(\vec{a}+\vec{c})]\times(\vec{b}+\vec{c})\}\cdot(\vec{b}+\vec{c})$ is

(a) $|\vec{a}|^2$ (b) $2|\vec{a}|^2$ (c) $3|\vec{a}|^2$ (d) 0

7. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=-\hat{i}+\hat{j}+\hat{k}$, $\vec{c}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{d}=\hat{i}+\hat{j}-\hat{k}$. Then, the line of intersection of planes one determined by $\vec{a}$, $\vec{b}$ and other determined by $\vec{c}$, $\vec{d}$ is perpendicular to

(a) x-axis (b) y-axis
(c) both x and y-axes (d) both y and z-axes

8. If a line with direction ratios (2, 2, 1) intersects the line $\frac{x-7}{3}=\frac{y-5}{2}=\frac{z-3}{1}$ and $\frac{x-1}{2}=\frac{y+1}{4}=\frac{z+1}{3}$ at A and B respectively, then AB =

(a) $\sqrt{2}$ (b) 2 (c) $\sqrt{3}$ (d) 3

9. Equation of the plane containing the straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$ is

(a) $x+2y-2z=0$ (b) $3x+2y-2z=0$
(c) $x-2y+z=0$ (d) $5x+2y-4z=0$

10. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-coplanar vectors and $\vec{d}$ be a non-zero vector, which is perpendicular to $\vec{a}+\vec{b}+\vec{c}$. Now, if $\vec{d}=(\sin x)(\vec{a}\times\vec{b})+(\cos y)(\vec{b}\times\vec{c})+2(\vec{c}\times\vec{a})$, then minimum value of x^2+y^2 is

(a) π^2 (b) $\frac{\pi^2}{2}$ (c) $\frac{\pi^2}{4}$ (d) $\frac{5\pi^2}{4}$

Multiple Correct Answer Type

11. A vector of magnitude 2 along a bisector of the angle between the two vectors $2\hat{i}-2\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}-2\hat{k}$ is

(a) $\frac{2}{\sqrt{10}}(3\hat{i}-\hat{k})$ (b) $\frac{1}{\sqrt{26}}(\hat{i}-4\hat{i}+3\hat{k})$

(c) $\frac{2}{\sqrt{26}}(\hat{i}-4\hat{i}+3\hat{k})$ (d) $\frac{2}{\sqrt{5}}(\hat{i}-3\hat{k})$

12. The plane $lx+my=0$ is rotated about its line of intersection with the plane $z=0$, through an angle α, then equation of plane in its new position may be

(a) $lx+my+z\sqrt{l^2+m^2}\tan\alpha=0$

(b) $lx+my-z\sqrt{l^2+m^2}\tan\alpha=0$

(c) data is not sufficient

(d) none of these

13. Let a plane passes through origin and is parallel to the line $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z+1}{-2}$ such that distance between plane and the line is 5/3. Then equation of the plane is

(a) $x-2y+2z=0$ (b) $x-2y-2z=0$
(c) $2x+2y+z=0$ (d) $x+y+z=0$

14. If x and y are two unit vectors and ϕ is the angle between them, then $\frac{1}{2}|x-y|$ is equal to

(a) 0 (b) $\frac{1}{\sqrt{2}}\sqrt{1-\cos\phi}$

(c) $\left|\sin\frac{\phi}{2}\right|$ (d) $\left|\cos\frac{\phi}{2}\right|$

15. If the median through A of ΔABC having vertices $A\equiv(2,3,5)$, $B\equiv(-1,3,2)$ and $C\equiv(\lambda,5,\mu)$ is equally inclined to the axes, then

(a) $\lambda=7$ (b) $\mu=10$ (c) $\lambda=10$ (d) $\mu=7$

16. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-zero vectors, then which of the following statement(s) is/are true ?

(a) $\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}\times(\vec{c}\times\vec{a})$, $\vec{c}\times(\vec{a}\times\vec{b})$ form a right handed system.

(b) $\vec{c},(\vec{a}\times\vec{b})\times\vec{c}$, $\vec{a}\times\vec{b}$ form a right handed system.

(c) $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}<0$ if $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

(d) $\frac{(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})}{(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{c})}=-1$ if $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

17. The plane $x-2y+7z+21=0$

(a) contains the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$.

(b) contains the point (0, 7, –1).

(c) is perpendicular to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{7}$.

(d) is parallel to the plane $x-2y+7z=0$.

18. If $\vec{a},\vec{b},\vec{c}$ are non-coplanar, non-zero vectors and $\vec{r}$ is any vector in space, then

$[\vec{b}\,\vec{c}\,\vec{r}]\vec{a}+[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}+[\vec{a}\,\vec{b}\,\vec{r}]\vec{c}$ is equal to

(a) $3[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$ (b) $[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$
(c) $[\vec{b}\,\vec{c}\,\vec{a}]\vec{r}$ (d) $6[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$

19. Let $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors. If $\vec{u}=\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}$ and $\vec{v}=\vec{a}\times\vec{b}$, then $|\vec{v}|=$

(a) $|\vec{u}|$ (b) $|\vec{u}|+|\vec{u}\cdot\vec{a}|$

(c) $|\vec{u}|+|\vec{u}\cdot\vec{b}|$ (d) $|\vec{u}|+\vec{u}\cdot(\vec{a}+\vec{b})$

Comprehension Type

Paragraph for Question No. 20 to 21

Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are forming a right handed system, if $\vec{a}\times\vec{b}=\vec{c}$, $\vec{b}\times\vec{c}=\vec{a}$, $\vec{c}\times\vec{a}=\vec{b}$.

20. If $\vec{x}=\vec{a}+\vec{b}-\vec{c}$, $\vec{y}=-\vec{a}+\vec{b}-2\vec{c}$, $\vec{z}=-\vec{a}+2\vec{b}-\vec{c}$, then a unit vector normal to the vector $\vec{x}+\vec{y}$ and $\vec{y}+\vec{z}$ is

(a) $\vec{a}$ (b) $\vec{b}$

(c) $\vec{c}$ (d) none of these

21. Let $\vec{x}=\vec{a}+\vec{b}$, $\vec{y}=2\vec{a}-\vec{b}$, then the point of intersection of straight lines $\vec{r}\times\vec{x}=\vec{y}\times\vec{x}$ and $\vec{r}\times\vec{y}=\vec{x}\times\vec{y}$ is

(a) $2\vec{b}$ (b) $3\vec{b}$

(c) $3\vec{a}$ (d) none of these

Paragraph for Question No. 22 to 23

Consider the equations of planes

$P_1\equiv\vec{r}\cdot(\hat{i}+2\hat{j}+\hat{k})-3=0,\ P_2\equiv\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})-5=0$

22. The equation of plane passing through the intersection of $P_1 = 0$, $P_2 = 0$ and through the point $A(1, 1, 1)$ is

(a) $\vec{r}\cdot(5\hat{i}-4\hat{j}+5\hat{k})=6$ (b) $\vec{r}\cdot(5\hat{i}+5\hat{j}-4\hat{k})=6$

(c) $\vec{r}\cdot(5\hat{i}+5\hat{j}+4\hat{k})=14$ (d) none of these

23. The line of intersection of planes $P_1 = 0$, $P_2 = 0$ is parallel to

(a) $3\hat{i}-5\hat{j}-\hat{k}$ (b) $3\hat{i}+\hat{j}-5\hat{k}$

(c) $2\hat{i}-\hat{j}-\hat{k}$ (d) none of these

Matrix-Match Type

24. Match the following :

	Column-I		Column-II
A.	The area of the triangle whose vertices are the points, with rectangular cartesian coordinates (1, 2, 3), (–2, 1, –4), (3, 4, –2) is	p.	0
B.	The value of $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})+(\vec{b}\times\vec{c})\cdot(\vec{a}\times d)+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})$ is	q.	$\frac{9}{2}$
C.	If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then value of k is	r.	$\frac{\sqrt{1218}}{2}$
D.	$\vec{a}=2\hat{i}+3\hat{j}-\hat{k}, \vec{b}=-\hat{i}+2\hat{j}-4\hat{k}$, $\vec{c}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{d}=3\hat{i}+2\hat{j}+\hat{k}$ then $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=$	s.	21

25. Match the following :

	Column-I		Column-II
A.	The value of α for which the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}+\alpha\hat{k}$ and $3\hat{i}-4\hat{j}+5\hat{k}$ are coplanar is	p.	4
B.	The area of a parallelogram having adjacent sides $\vec{a}=3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}-3\hat{j}+4\hat{k}$ is	q.	–3
C.	If $\vec{r}\cdot\vec{a}=0, \vec{r}\cdot\vec{b}=0$ and $\vec{r}\cdot\vec{c}=0$ for some non-zero vector $\vec{r}$, then the value of $[\vec{a}\ \vec{b}\ \vec{c}]$ is	r.	$10\sqrt{3}$
D.	The volume of parallelepiped whose sides are given $\overrightarrow{OA}=2\hat{i}-3\hat{j}$, $\overrightarrow{OB}=\hat{i}+\hat{j}-\hat{k}$ and $\overrightarrow{OC}=3\hat{i}-\hat{k}$ is	s.	0

Numerical Answer Type

26. Let $\vec{c}$ be a unit vector coplanar with $\vec{a}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$ such that $\vec{c}$ is perpendicular to $\vec{a}$. If p be the projection of $\vec{c}$ along $\vec{b}$, then evaluate $\frac{\sqrt{11}}{p}$.

27. Find the distance of the point $\hat{i}+2\hat{j}+3\hat{k}$ from the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=5$ measured parallel to the vector $2\hat{i}+3\hat{j}-6\hat{k}$.

28. If $\vec{a}, \vec{b}, \vec{c}$ are the three unit vectors and α, β, γ are scalars such that $\vec{c}=\alpha\vec{a}+\beta\vec{b}+\gamma(\vec{a}\times\vec{b})$. It is given that $\vec{a}\cdot\vec{b}=0$ and $\vec{c}$ makes equal angle with both $\vec{a}$ and $\vec{b}$, then evaluate $\alpha^2+\beta^2+\gamma^2$.

29. The equation of the plane passing through the intersection of the planes $2x - 5y + z = 3$ and $x + y + 4z = 5$ and parallel to the plane $x + 3y + 6z = 1$ is $x + 3y + 6z = k$, where k is _____.

30. If $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$ and $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=\frac{1}{2}$, then the value of $4\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2$ is _____.

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SOLUTIONS

1. (a) : $\vec{\alpha}\cdot\vec{\beta}=\frac{b}{a}+\frac{4a}{b}+1$ as $\frac{b}{a}+\frac{4a}{b}+1\geq 5$

So, $\left(\frac{10}{5+\vec{\alpha}\cdot\vec{\beta}}\right)_{\max}=1$

2. (b) : Equation of the plane bisecting the angle between the given planes are

$$\frac{2x-y+2z+3}{\sqrt{(2)^2+(-1)^2+2^2}}=\pm\frac{3x-2y+6z+8}{\sqrt{3^2+(-2)^2+6^2}}$$

$\Rightarrow$ $7(2x - y + 2z + 3) = \pm 3(3x - 2y + 6z + 8)$

$\Rightarrow$ $5x - y - 4z - 3 = 0$ taking the +ve sign, and

$23x - 13y + 32z + 45 = 0$ taking the – ve sign.

3. (c) : Since the line makes an angle θ with the plane. So, it makes an angle $\pi/2 - \theta$ with normal to the plane

$$\therefore\ \cos\left(\frac{\pi}{2}-\theta\right)=\frac{2(1)+(-1)(2)+(\sqrt{\lambda})(2)}{\sqrt{1+4+4}\times\sqrt{4+1+\lambda}}$$

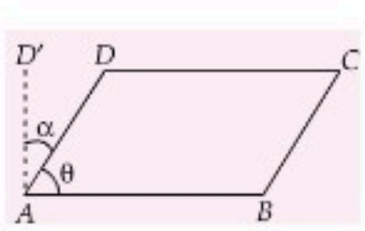
$$\Rightarrow\ \frac{1}{3}=\frac{2\sqrt{\lambda}}{3\sqrt{\lambda+5}}\Rightarrow\lambda+5=4\lambda\ \Rightarrow\ \lambda=5/3$$

4. (b) : Since, $\cos\theta=\frac{\overrightarrow{AB}\cdot\overrightarrow{AD}}{|\overrightarrow{AB}||\overrightarrow{AD}|}=\frac{8}{9}$

$\therefore\ \cos^{-1}\left(\frac{8}{9}\right)+\alpha=\frac{\pi}{2}$ (by hypothesis)

$\therefore\ \sin\alpha=\frac{8}{9}$

$\Rightarrow\ \cos\alpha=\sqrt{1-\frac{64}{81}}=\frac{\sqrt{17}}{9}$



5. (a) : Equation of any plane parallel to the given plane is $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+\lambda=0$.

Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then equation of plane becomes

$2x - 3y + 5z + \lambda = 0$

This plane passes through the point (3, 4, –1).

$\therefore\ 2\times 3 - 3\times 4 + 5(-1) + \lambda = 0 \Rightarrow \lambda = 11$

Hence the equation of the required plane is

$\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+11=0$

6. (d)

7. (d) : $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\ \vec{b}\ \vec{d}]\vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

$=-4\vec{c}-4\vec{d}=-8\hat{i}$

Which is perpendicular to both y and z-axes.

8. (d) : Coordinates of A and B are

$(7 + 3\alpha, 5 + 2\alpha, 3 + \alpha)$ and $(1 + 2\beta, -1 + 4\beta, -1 + 3\beta)$ respectively.

d.r.'s of AB are (2, 2, 1)

$\therefore$ Equation of AB is

$$\frac{6+3\alpha-2\beta}{2}=\frac{3+\alpha-2\beta}{1}=\frac{4+\alpha-3\beta}{1}$$

$\Rightarrow$ $\alpha = -2, \beta = 1$

$\therefore$ $A(1,1,1)$, $B(3,3,2)$, $AB = 3$

9. (c) : Vector along the required plane is

$$\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&4&2\\4&2&3\end{vmatrix}=8\hat{i}-\hat{j}-10\hat{k}$$

So, normal vector $(\vec{n})$ to the plane is

$$\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\8&-1&-10\\2&3&4\end{vmatrix}=26\hat{i}-52\hat{j}+26\hat{k}.$$

So, equation of the plane is $\vec{r}\cdot\vec{n}=0\Rightarrow x-2y+z=0.$

10. (d) : Since $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar $\Rightarrow$ $[\vec{a}, \vec{b}, \vec{c}]\neq 0$

Also, $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ are non-coplanar.

Given $\vec{d}=(\sin x)(\vec{a}\times\vec{b})+(\cos y)(\vec{b}\times\vec{c})+2(\vec{c}\times\vec{a})$.

Taking dot product with $\vec{a}+\vec{b}+\vec{c}$, we get

$0=\sin x\,[\vec{a}\ \vec{b}\ \vec{c}]+\cos y[\vec{a}\ \vec{b}\ \vec{c}]+2[\vec{a}\ \vec{b}\ \vec{c}]$

$\Rightarrow$ $\sin x + \cos y + 2 = 0$ $\Rightarrow$ $\sin x + \cos y = -2$

$\Rightarrow\ x=(4n-1)\frac{\pi}{2};\ \ y=(2n+1)\pi,\ n\in Z$

For least value of $x^2 + y^2$, $x=\frac{-\pi}{2}$, $y=\pi$

and least value is $\frac{5\pi^2}{4}$.

11. (a, c) : $\vec{a}=2\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

$\therefore$ $|\vec{a}| = 3$, $|\vec{b}| = 3$

$\therefore$ A vector along bisectors

$$=\frac{\vec{a}}{|\vec{a}|}\pm\frac{\vec{b}}{|\vec{b}|}=\hat{i}-\frac{1}{3}\hat{k},\frac{1}{3}\hat{i}-\frac{4}{3}\hat{j}+\hat{k}$$

$\therefore$ The required vector

$$=2.\frac{\hat{i}-\frac{1}{3}\hat{k}}{\sqrt{1^2+\left(-\frac{1}{3}\right)^2}},2.\frac{\frac{1}{3}\hat{i}-\frac{4}{3}\hat{j}+\hat{k}}{\sqrt{\left(\frac{1}{3}\right)^2+\left(-\frac{4}{3}\right)^2+1^2}}$$

$$=\frac{2}{\sqrt{10}}(3\hat{i}-\hat{k}),\frac{2}{\sqrt{26}}(\hat{i}-4\hat{j}+3\hat{k})$$

12. (a, b) : Let equation of required plane is

$lx + my + \lambda z = 0$ ...(1)

Angle between (1) and $lx + my = 0$ is α.

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$$\Rightarrow \quad \cos\alpha = \frac{l^2+m^2}{\sqrt{l^2+m^2}\sqrt{l^2+m^2+\lambda^2}}$$

$$\Rightarrow \cos^2\alpha = \frac{l^2+m^2}{l^2+m^2+\lambda^2} \Rightarrow \lambda = \pm\sqrt{l^2+m^2}\tan\alpha$$

Hence, equation of plane is

$lx+my \pm z\sqrt{l^2+m^2}\tan\alpha = 0$

13. (a, c) : Let the equation of plane be

$lx + my + nz = 0$, where l, m, n be d.c.'s

$\Rightarrow$ $l^2 + m^2 + n^2 = 1$ ...(i)

Given line is $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z+1}{-2}$

$\therefore$ $2l - m - 2n = 0$ ...(ii)

Also, $\frac{l-3m-n}{\sqrt{l^2+m^2+n^2}} = \frac{5}{3} \Rightarrow l-3m-n = \frac{5}{3}$ ...(iii)

Solving (i), (ii) and (iii) we get equation of plane as

$x - 2y + 2z = 0$ or $2x + 2y + z = 0$.

14. (b, c) : $|x - y|^2 = |x|^2 + |y|^2 - 2xy = 1 + 1 - 2\cos\phi$

$= 2(1 - \cos\phi)$

$\Rightarrow$ $|x - y| = \sqrt{2}\sqrt{1-\cos\phi}$

$\Rightarrow$ $\frac{1}{2}|x-y| = \frac{1}{\sqrt{2}}\sqrt{1-\cos\phi} = \left|\sin\frac{\phi}{2}\right|$

15. (a, b) : Mid-point of $BC = \left(\frac{\lambda-1}{2}, 4, \frac{\mu+2}{2}\right)$

d.r.'s of median through A are

$\left(\frac{\lambda-1}{2}-2, 4-3, \frac{\mu+2}{2}-5\right) = \left(\frac{\lambda-5}{2}, 1, \frac{\mu-8}{2}\right)$

The median is equally inclined to axes, so the direction ratios must be equal, so

$\frac{\lambda-5}{2} = 1 = \frac{\mu-8}{2} \Rightarrow \lambda = 7, \mu = 10$

16. (b, c, d) : (a) $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$

$\Rightarrow$ vectors are coplanar, so do not form RHS.

(b) $(\vec{a}\times\vec{b})\times\vec{c}$, $\vec{a}\times\vec{b}$, $\vec{c}$ in this order form RHS.

$\Rightarrow$ $\vec{c}, (\vec{a}\times\vec{b})\times\vec{c}, \vec{a}\times\vec{b}$ also form RHS as they are in same cyclic order.

(c) $\vec{a}+\vec{b}+\vec{c} = \vec{0} \Rightarrow (\vec{a}+\vec{b}+\vec{c})^2 = 0$

$\Rightarrow \vec{a}^2+\vec{b}^2+\vec{c}^2 = -2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$

Hence, $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} < 0$

(d) $\vec{a}+\vec{b}+\vec{c} = \vec{0} \Rightarrow \vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

Using this we get the result.

17. (a, b, c, d) : Conditions for the plane $ax+by+cz+d=0$ to contain the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ are $a\alpha + b\beta + c\gamma + d = 0$ and $al + bm + cn = 0$.

Here the condition is satisfied for option (a).

For option (c), normal to plane and given line have same d.r.'s.

$\therefore$ The line is perpendicular to the given plane.

For option (b), clearly the point satisfies the given plane.

The d.r.'s of normal to the plane in option (d) are $(1, -2, 7)$ which are same as the d.r.'s of normal to the given plane.

$\therefore$ The plane given in option (d) is parallel to given plane.

18. (b, c)

19. (a, c) : Given $\vec{v} = \vec{a}\times\vec{b} \Rightarrow |\vec{v}| = |\vec{a}||\vec{b}|\sin\theta = \sin\theta$

$\vec{u} = \vec{a} - (\vec{a}\cdot\vec{b})\vec{b} = \vec{a} - \vec{b}\cos\theta$

$\Rightarrow |\vec{u}|^2 = (\vec{a}-\vec{b}\cos\theta)^2 = |\vec{a}|^2 + |\vec{b}|^2\cos^2\theta - 2\vec{a}\cdot\vec{b}\cos\theta$

$= 1 + \cos^2\theta - 2\cos^2\theta = 1 - \cos^2\theta = \sin^2\theta \Rightarrow |\vec{u}| = |\vec{v}|$

Again $\vec{u}\cdot\vec{b} = \vec{a}\cdot\vec{b} - (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{b}) = 0 \Rightarrow |\vec{u}\cdot\vec{b}| = 0$

So, $|\vec{v}| = |\vec{u}| + |\vec{u}\cdot\vec{b}|$

20. (d) : $\vec{x}+\vec{y} = 2\vec{b} - 3\vec{c}$ and $\vec{y}+\vec{z} = -2\vec{a} + 3\vec{b} - 3\vec{c}$

$\therefore$ $(\vec{x}+\vec{y})\times(\vec{y}+\vec{z}) = \begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ 0 & 2 & -3 \\ -2 & 3 & -3 \end{vmatrix} = 3\vec{a} + 6\vec{b} + 4\vec{c}$

$\therefore$ Required unit vector $= \frac{3\vec{a}+6\vec{b}+4\vec{c}}{\sqrt{61}}$

21. (c) : $\vec{r}\times\vec{x} = \vec{y}\times\vec{x} \Rightarrow (\vec{r}-\vec{y})\times\vec{x} = 0 \Rightarrow \vec{r} = \vec{y} + \lambda\vec{x}$

$\vec{r}\times\vec{y} = \vec{x}\times\vec{y} \Rightarrow (\vec{r}-\vec{x})\times\vec{y} = 0 \Rightarrow \vec{r} = \vec{x} + \mu\vec{y}$

$\Rightarrow$ $\vec{y} + \lambda\vec{x} = \vec{x} + \mu\vec{y}$

$\Rightarrow$ $(2\vec{a} - \vec{b}) + \lambda(\vec{a}+\vec{b}) = (\vec{a}+\vec{b}) + \mu(2\vec{a}-\vec{b})$

$\Rightarrow$ $2 + \lambda = 1 + 2\mu$, $-1 + \lambda = 1 - \mu \Rightarrow \mu = 1, \lambda = 1$

$\therefore$ The point of intersection is $3\vec{a}$.

22. (c) : The required plane is

$x + 2y + z - 3 + k(2x - y + z - 5) = 0$

Since it passes through A (1, 1, 1).

$\therefore$ $k = 1/3$

$\therefore$ The equation of plane is $5x + 5y + 4z - 14 = 0$,

i.e., $\vec{r}\cdot(5\hat{i}+5\hat{j}+4\hat{k}) = 14$

23. (b) : The line of intersection of planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is parallel to $\vec{n}_1\times\vec{n}_2$.

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24. A → r; B → p; C → q; D → s

(A) $\overrightarrow{OA}=\hat{i}+2\hat{j}+3\hat{k},\ \overrightarrow{OB}=-2\hat{i}+\hat{j}-4\hat{k},$
$\overrightarrow{OC}=3\hat{i}+4\hat{j}-2\hat{k}$

$\therefore\quad \text{Area}=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{\sqrt{1218}}{2}$

(B) $((\vec{a}\times\vec{b})\times\vec{c})\cdot\vec{d}+((\vec{b}\times\vec{c})\times\vec{a})\cdot\vec{d}+((\vec{c}\times\vec{a})\times\vec{b})\cdot\vec{d}=0$

(C) Since, the given lines intersect.

$\therefore\ (2r+1,3r-1,4r+1)=(s+3,2s+k,s)$

$\Rightarrow\ r=\frac{-3}{2},s=-5\ \Rightarrow\ k=\frac{9}{2}$

(D) Here, $\vec{a}\times\vec{b}=-10\hat{i}+9\hat{j}+7\hat{k}$

and $\vec{c}\times\vec{d}=-\hat{i}+2\hat{j}-\hat{k}$

$\therefore\ (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=10+18-7=21$

25. A → q; B → r; C → s; D → p

(A) Since, vectors are coplanar

$\therefore\quad \begin{vmatrix}2&-1&1\\1&2&\alpha\\3&-4&5\end{vmatrix}=0\ \Rightarrow \alpha=-3$

(B) Area of parallelogram $=\left|\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&1&-2\\1&-3&4\end{vmatrix}\right|=10\sqrt{3}$

(C) Since, $\vec{a},\vec{b},\vec{c}$ are coplanar. $\therefore\ [\vec{a}\ \vec{b}\ \vec{c}]=0$

(D) Volume of parallelepiped $=\begin{vmatrix}2&-3&0\\1&1&-1\\3&0&-1\end{vmatrix}=4$

26. (6) : Let $\vec{c}=x\vec{a}+y\vec{b}$, where x, y are scalars.

$\Rightarrow\ \vec{c}=x(\hat{i}-\hat{j}+2\hat{k})+y(2\hat{i}-\hat{j}+\hat{k})$

$\Rightarrow\ \vec{c}=\hat{i}(x+2y)+\hat{j}(-x-y)+\hat{k}(2x+y)$

But $\vec{c}\cdot\vec{a}=0\Rightarrow 6x+5y=0\Rightarrow y=\frac{-6x}{5}$

So, $\vec{c}=\frac{-7x}{5}\hat{i}+\frac{x}{5}\hat{j}+\frac{4x}{5}\hat{k}$

We have, $\frac{49x^2+x^2+16x^2}{25}=1\ \Rightarrow x^2=\frac{25}{66}$

$\therefore\ \vec{c}=\pm\frac{5}{\sqrt{66}}\left(\frac{-7}{5}\hat{i}+\frac{1}{5}\hat{j}+\frac{4}{5}\hat{k}\right);\ p=\frac{|\vec{c}\cdot\vec{b}|}{|\vec{b}|}=\frac{\sqrt{11}}{6}$

So, $\frac{\sqrt{11}}{p}=6$

27. (7) : The distance of the point 'a' from the plane $\vec{r}\cdot\vec{n}=q$ measured in the direction of the unit vector $\vec{b}=\frac{q-\vec{a}\cdot\vec{n}}{\hat{b}\cdot\vec{n}}$

Here, $\vec{a}=\hat{i}+2\hat{j}+3\hat{k},\ \vec{n}=\hat{i}+\hat{j}+\hat{k}$ and $q=5$

Also, $\hat{b}=\frac{2\hat{i}+3\hat{j}-6\hat{k}}{\sqrt{(2)^2+(3)^2+(-6)^2}}=\frac{2\hat{i}+3\hat{j}-6\hat{k}}{7}$

$\therefore$ The required distance

$=\left|\frac{5-(\hat{i}+2\hat{j}+3\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})}{\frac{1}{7}(2\hat{i}+3\hat{j}-6\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})}\right|=\left|\frac{5-(1+2+3)}{\frac{1}{7}(2+3-6)}\right|=7$

28. (1) : Let $\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}=\cos\theta$ and

also $\vec{c}=\alpha\vec{a}+\beta\vec{b}+\gamma(\vec{a}\times\vec{b})$

Taking dot product with $\vec{a}$ both sides, we get $\cos\theta=\alpha$

Taking dot product with $\vec{b}$ both sides, we get $\cos\theta=\beta$

Taking dot product with $\vec{c}$ both sides, we get

$1=\alpha\cos\theta+\beta\cos\theta+\gamma[\vec{a}\ \vec{b}\ \vec{c}]$

But $[\vec{a}\,\vec{b}\,\vec{c}]^2=\begin{vmatrix}1&0&\cos\theta\\0&1&\cos\theta\\\cos\theta&\cos\theta&1\end{vmatrix}=1-2\cos^2\theta$

So, $1=\cos^2\theta+\cos^2\theta+\gamma\sqrt{1-2\cos^2\theta}$

$\Rightarrow\ \gamma=\sqrt{1-2\cos^2\theta}$

So, $\alpha^2+\beta^2+\gamma^2=1$

29. (7) : Equation of plane passing through the intersection of the planes $2x-5y+z=3$ and $x+y+4z=5$ is

$(2x-5y+z-3)+\lambda(x+y+4z-5)=0$

$\Rightarrow(2+\lambda)x+(-5+\lambda)y+(1+4\lambda)z-3-5\lambda=0$ ....(i)

which is parallel to the plane $x+3y+6z=1$.

Then, $\frac{2+\lambda}{1}=\frac{-5+\lambda}{3}=\frac{1+4\lambda}{6}\ \therefore\lambda=\frac{-11}{2}$

From (i), we have, $-\frac{7}{2}x-\frac{21}{2}y-21z+\frac{49}{2}=0$

$\therefore\quad x+3y+6z=7$. Hence, $k=7$

30. (2) : $[\vec{a}\ \vec{b}\ \vec{c}]^2=[\vec{a}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}]$

$=\begin{vmatrix}\vec{a}\cdot\vec{a}&\vec{a}\cdot\vec{b}&\vec{a}\cdot\vec{c}\\\vec{b}\cdot\vec{a}&\vec{b}\cdot\vec{b}&\vec{b}\cdot\vec{c}\\\vec{c}\cdot\vec{a}&\vec{c}\cdot\vec{b}&\vec{c}\cdot\vec{c}\end{vmatrix}=\begin{vmatrix}1&1/2&1/2\\1/2&1&1/2\\1/2&1/2&1\end{vmatrix}=\frac{1}{2}$

$\Rightarrow\ 4[\vec{a}\ \vec{b}\,\vec{c}]^2=2$

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CBSE warm-up!

CLASS-XII

Chapterwise Practice questions for CBSE Exams as per the reduced syllabus and marking scheme issued by CBSE for the academic session 2020-21.

Series 7 — **Vectors**

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(1) This question paper contains two parts A and B. Each part is compulsory. Part A carries 24 marks and Part B carries 56 marks.
(2) Part-A has Objective Type Questions and Part -B has Descriptive Type Questions.
(3) Both Part A and Part B have internal choices.

Part-A :

(1) It consists of two sections- I and II.
(2) Section I comprises of 16 very short answer type questions.
(3) Section II contains 2 case studies. Each case study comprises of 5 case-based MCQs. An examinee is to attempt any 4 out of 5 MCQs.

Part-B :

(1) It consists of three sections- III, IV and V.
(2) Section III comprises of 10 questions of 2 marks each.
(3) Section IV comprises of 7 questions of 3 marks each.
(4) Section V comprises of 3 questions of 5 marks each.
(5) Internal choice is provided in 3 questions of Section-III, 2 questions of Section-IV and 3 questions of Section-V. You have to attempt only one of the alternatives in all such questions.

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PART-A

SECTION-I

All questions are compulsory. In case of internal choices attempt any one.

1. Find the magnitude of the vector $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$.

OR

Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

2. If A, B and C are the vertices of a triangle ABC, then what is the value of $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$?

3. Find a vector in the direction of $\vec{a} = \hat{i} - 2\hat{j}$ that has magnitude 7 units.

OR

If $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$.

4. If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then write the value of $x + y + z$.

5. Find the scalar components of the vector $\overrightarrow{AB}$ with initial point $A(2, 1)$ and terminal point $B(-5, 7)$.

OR

If vectors $\vec{a}$ and $\vec{b}$ are such that, $|\vec{a}|=3, |\vec{b}|=\frac{2}{3}$ and $\vec{a}\times\vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$.

6. Find a unit vector in the direction of $\vec{a}=3\hat{i}-2\hat{j}+6\hat{k}$.

7. For what value of 'a', the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $a\hat{i}+6\hat{j}-8\hat{k}$ are collinear ?

OR

If $|\vec{a}|=\sqrt{3}, |\vec{b}|=2$ and angle between $\vec{a}$ and $\vec{b}$ is 60°, find $\vec{a}\cdot\vec{b}$.

8. Write the value of $(\hat{k}\times\hat{j})\cdot\hat{i}+\hat{j}\cdot\hat{k}$.

9. If $\left(\frac{1}{2}, \frac{1}{3}, n\right)$ are the direction cosines of a line, then find the value of n.

OR

If O is origin and C is the mid point of $A(2, -1)$ and $B(-4, 3)$, then find the value of $\overrightarrow{OC}$.

10. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors, then find $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})$.

11. Find the value of λ so that the vectors $2\hat{i}-4\hat{j}+\hat{k}$ and $4\hat{i}-8\hat{j}+\lambda\hat{k}$ are perpendicular.

12. If $\overrightarrow{AB}\times\overrightarrow{AC}=2\hat{i}-4\hat{j}+4\hat{k}$, then find the area of ΔABC.

13. If the two vectors $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=4\hat{i}-\lambda\hat{j}+6\hat{k}$ are parallel, then find the value of λ.

14. If $|\vec{a}|=5, |\vec{b}|=13$ and $|\vec{a}\times\vec{b}|=25$, then find $\vec{a}\cdot\vec{b}$.

15. If position vector of a point A is $\vec{a}+2\vec{b}$ and any point $P(\vec{a})$ divides $\overrightarrow{AB}$ in the ratio of 2 : 3, then find the position vector of B.

16. Write the projection of the vector $\hat{i}+\hat{j}+\hat{k}$ along the vector $\hat{j}$.

SECTION - II

Both the case study based questions are compulsory. Attempt any 4 sub-parts from each question 17 and 18. Each sub-part carries 1 mark.

17. If two vectors are represented by the two sides of a triangle taken in order, then their sum is represented by the third side of the triangle taken in opposite order.

(i) If $\vec{p}, \vec{q}, \vec{r}$ are the vectors represented by the sides of a triangle taken in orders, then $\vec{p}+\vec{q}=$

(a) $\vec{r}$ (b) $2\vec{r}$

(c) $-\vec{r}$ (d) None of these

(ii) $ABCD$ is a parallelogram. If AC and BD are its diagonals, then $\overrightarrow{AC}+\overrightarrow{BD}=$

(a) $2\overrightarrow{AD}$ (b) $2\overrightarrow{AB}$

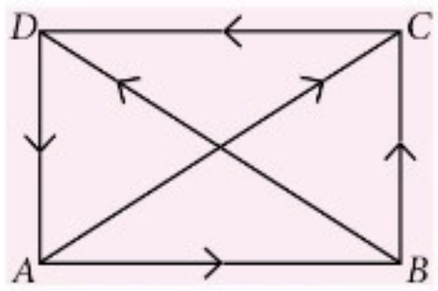
(c) $2\overrightarrow{BC}$ (d) $2\overrightarrow{BD}$

(iii) If $ABCD$ is a parallelogram, where $\overrightarrow{AB}=\vec{x}$ and $\overrightarrow{BC}=\vec{y}$, then $\overrightarrow{AC}-\overrightarrow{BD}=$

(a) $\vec{x}$ (b) $2\vec{x}$

(c) $\vec{y}$ (d) $2\vec{y}$

(iv) If $ABCD$ is a quadrilateral whose diagonals are $\overrightarrow{AC}$ and $\overrightarrow{BD}$, then $\overrightarrow{BA}+\overrightarrow{CD}=$



(a) $\overrightarrow{AC}+\overrightarrow{DB}$ (b) $\overrightarrow{AC}+\overrightarrow{BD}$

(c) $\overrightarrow{BC}+\overrightarrow{AD}$ (d) $\overrightarrow{BD}+\overrightarrow{CA}$

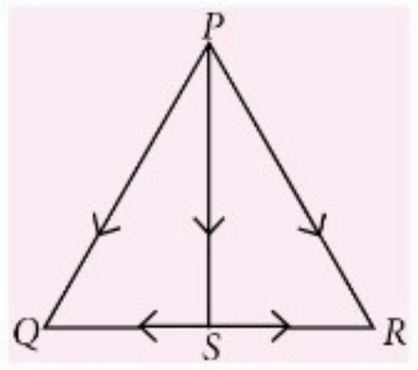
(v) If S is the mid point of side QR, of ΔPQR, then $\overrightarrow{PQ}+\overrightarrow{PR}=$

(a) $2\overrightarrow{QS}$

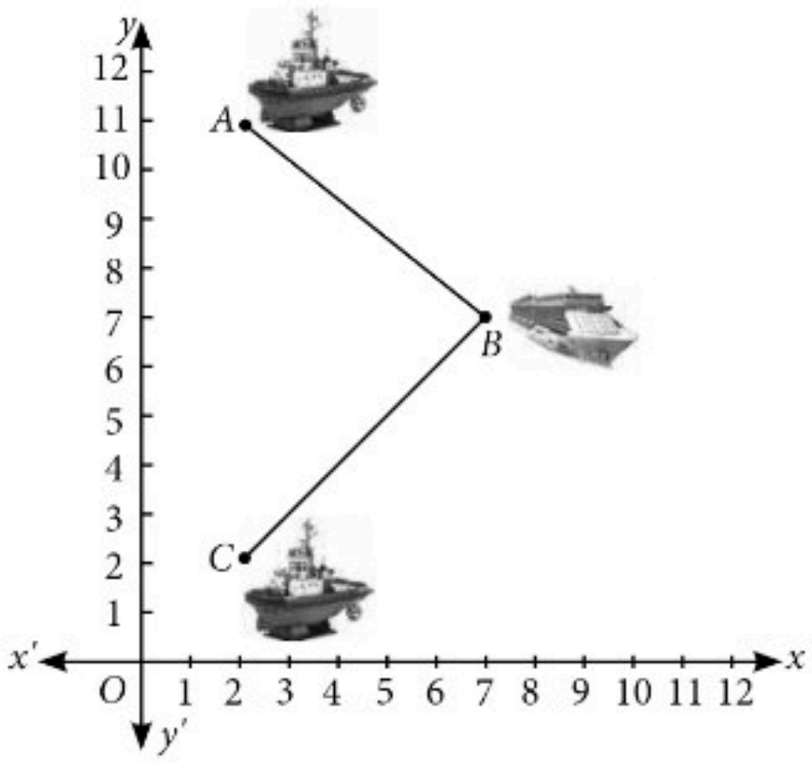
(b) $2\overrightarrow{PS}$

(c) $2\overrightarrow{SR}$

(d) None of these



18. A ship is pulled into harbour by two tug boats as shown in the figure.



Based on the above answer the following :

(i) Position vector of A is

(a) $11\hat{i}+2\hat{j}$ (b) $2\hat{i}+11\hat{j}$

(c) $2\hat{i}-11\hat{j}$ (d) $11\hat{i}-2\hat{j}$

(ii) Position vector of B is

(a) $4\hat{i}+4\hat{j}$ (b) $6\hat{i}+6\hat{j}$

(c) $7\hat{i}+7\hat{j}$ (d) $3\hat{i}+3\hat{j}$

(iii) Find the vector $\overline{AC}$ in terms of $\hat{i}, \hat{j}$.

(a) $9\hat{j}$ (b) $-9\hat{j}$

(c) $9\hat{i}$ (d) None of these

(iv) If $\vec{A}=\hat{i}+2\hat{j}+3\hat{k}$, then its unit vector is

(a) $\frac{1\hat{i}}{\sqrt{14}}+\frac{2\hat{j}}{\sqrt{14}}+\frac{3\hat{k}}{\sqrt{14}}$

(b) $\frac{3\hat{i}}{\sqrt{14}}+\frac{2\hat{j}}{\sqrt{14}}+\frac{\hat{k}}{\sqrt{14}}$

(c) $\frac{2\hat{i}}{\sqrt{14}}+\frac{3\hat{j}}{\sqrt{14}}+\frac{\hat{k}}{\sqrt{14}}$

(d) None of these

(v) If $\vec{A}=2\hat{i}+3\hat{j}$ and $\vec{B}=3\hat{i}+2\hat{j}$, then $|\vec{A}|=|\vec{B}|$ = _____.

(a) $\sqrt{5}$ (b) $\sqrt{7}$ (c) $\sqrt{11}$ (d) $\sqrt{13}$

PART-B

SECTION-III

All questions are compulsory. In case of internal choices attempt any one.

19. Let $\vec{a}=\hat{i}+\hat{j}-\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\hat{c}$ be a unit vector perpendicular to $\vec{a}$ and coplanar with $\vec{a}$ and $\vec{b}$, then find the vector $\hat{c}$.

20. The vector $\hat{i}+x\hat{j}+3\hat{k}$ is rotated through an angle θ and doubled in magnitude, then it becomes $4\hat{i}+(4x-2)\hat{j}+2\hat{k}$. Find the value of x.

OR

Find the length of longer diagonal of the parallelogram constructed on $5\vec{a}+2\vec{b}$ and $\vec{a}-3\vec{b}$. If it is given that $|\vec{a}|=2\sqrt{2}$, $|\vec{b}|=3$ and angle between $\vec{a}$ and $\vec{b}$ is $\pi/4$.

21. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+3\hat{j}+5\hat{k}$ and $\vec{c}=7\hat{i}+9\hat{j}+11\hat{k}$, then find the area of parallelogram having diagonal $\vec{a}+\vec{b}$ and $\vec{b}+\vec{c}$ (in sq. units).

22. If $\vec{a}=2\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}-2\hat{j}-\hat{k}$, $\vec{c}=\hat{i}+\hat{j}+\hat{k}$, then find $\vec{a}\times(\vec{b}\times\vec{c})$.

23. X and Y are two points with position vectors $3\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$ respectively. Write the position vector of a point Z which divides the line segment XY in the ratio 2 : 1 externally.

OR

Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $3\hat{i}+7\hat{j}+\hat{k}$ and $5\hat{i}+6\hat{j}+2\hat{k}$ form the sides of a right-angled triangle.

24. Prove that, for any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$ (Triangle inequality).

25. Let $\vec{a}, \vec{b}, \vec{c}$ be unit vectors such that $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}=0$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$, prove that $\vec{a}=\pm 2(\vec{b}\times\vec{c})$.

26. If with reference to the right handed system of mutually perpendicular unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$, $\vec{\alpha}=3\hat{i}-\hat{j}$, $\vec{\beta}=2\hat{i}+\hat{j}-3\hat{k}$, then express $\vec{\beta}$ in the form $\vec{\beta}=\vec{\beta_1}+\vec{\beta_2}$, where $\vec{\beta_1}$ is parallel to $\vec{\alpha}$ and $\vec{\beta_2}$ is perpendicular to $\vec{\alpha}$.

27. If $\vec{a}=5\hat{i}-\hat{j}-3\hat{k}$ and $\vec{b}=\hat{i}+3\hat{j}-5\hat{k}$, then show that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular to each other.

28. If $\vec{q}=-6\hat{i}+2\hat{j}-3\hat{k}$ $\vec{q}=-6\hat{i}+2\hat{j}-3\hat{k}$ then find $\vec{p}\times\vec{q}$. Verify that $\vec{p}$ and $\vec{p}\times\vec{q}$ are perpendicular to each other.

OR

Find a unit vector in the direction of $(\vec{a}+\vec{b})$, where $\vec{a}=-\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}-3\hat{k}$.

SECTION-IV

All questions are compulsory. In case of internal choices attempt any one.

29. Find the value of p for which the vectors $\vec{a}=3\hat{i}+2\hat{j}+9\hat{k}$ and $\vec{b}=\hat{i}+p\hat{j}+3\hat{k}$ are

(i) perpendicular (ii) parallel

30. Find the area of the triangle whose vertices are $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$.

31. Find a unit vector perpendicular to the plane ABC where A, B, C are the points $(3, -1, 2)$, $(1, -1, -3)$ and $(4, -3, 1)$ respectively.

OR

Given that $\vec{a}=\hat{i}+\hat{j}$, $\vec{b}=\hat{j}-3\hat{k}$ and $\vec{c}=\hat{i}+4\hat{k}$. Verify that $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$.

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32. The two vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represent the two sides AB and AC, respectively of a ΔABC. Find the length of the median through A.

33. Find the values of λ for which the angle between the vectors $\vec{a}=2\lambda^2\,\hat{i}+4\lambda\hat{j}+\hat{k}$ and $\vec{b}=7\hat{i}-2\hat{j}+\lambda\hat{k}$ is obtuse.

34. Define $\vec{a}\times\vec{b}$ and prove that $|\vec{a}\times\vec{b}|=(\vec{a}\cdot\vec{b})\tan\theta$, where θ is angle between $\vec{a}$ and $\vec{b}$.

OR

If $\vec{a}=2\hat{i}-\hat{j}+\hat{k}, \vec{b}=\hat{i}+3\hat{j}-\hat{k}, \vec{c}=-2\hat{i}+\hat{j}-3\hat{k}$ and $\vec{d}=3\hat{i}+2\hat{j}+5\hat{k}$, then find scalars α, β, γ such that $\vec{d}=\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}$.

35. If $\vec{a},\vec{b}$ and $\vec{c}$ are three vectors such that each one is perpendicular to the vector obtained by sum of the other two and $|\vec{a}|=3, |\vec{b}|=4$ and $|\vec{c}|=5$, then prove that $|\vec{a}+\vec{b}+\vec{c}|=5\sqrt{2}$.

SECTION - V

All questions are compulsory. In case of internal choices attempt any one.

36. If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}, 3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of points A, B, C and D respectively, then find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Deduce that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

OR

Express the vector $\vec{a}=2\hat{i}-\hat{j}+3\hat{k}$ as the sum of two vectors such that one is parallel to the vector $\vec{b}=2\hat{i}+4\hat{j}-2\hat{k}$ and other is perpendicular to $\vec{b}$.

37. If $\vec{a}$ and $\vec{b}$ are two unit vectors and θ is the angle between them, then show that $\cos\frac{\theta}{2}=\frac{1}{2}|\vec{a}+\vec{b}|$.

OR

Find a vector whose magnitude is 3 units and which is perpendicular to the following two vectors $\vec{a}$ and $\vec{b}$: $\vec{a}=3\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b}=6\hat{i}+5\hat{j}-2\hat{k}$.

38. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram.

OR

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$. Also, find the angle which $\vec{a}+\vec{b}+\vec{c}$ makes with $\vec{a}$ or $\vec{b}$ or $\vec{c}$.

SOLUTIONS

1. Here, $\vec{a}=3\hat{i}-2\hat{j}+6\hat{k}$

$\therefore$ Its magnitude $=|\vec{a}|$

$=\sqrt{3^2+(-2)^2+6^2}=\sqrt{9+4+36}=\sqrt{49}=7.$

OR

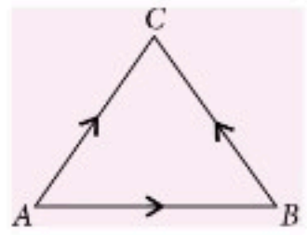
The given vectors are

$\vec{a}=\hat{i}-2\hat{j}+\hat{k}, \vec{b}=-2\hat{i}+4\hat{j}+5\hat{k}, \vec{c}=\hat{i}-6\hat{j}-7\hat{k}$

$\therefore$ Their sum $=\vec{a}+\vec{b}+\vec{c}$

$=(\hat{i}-2\hat{j}+\hat{k})+(-2\hat{i}+4\hat{j}+5\hat{k})+(\hat{i}-6\hat{j}-7\hat{k})=-4\hat{j}-\hat{k}.$

2. Let ABC be the given triangle.



Now $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

(By Triangle law)

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{AC}+\overrightarrow{CA}=\vec{0}$

3. A unit vector in the direction of $\vec{a}=\hat{i}-2\hat{j}$

is $\hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{\hat{i}-2\hat{j}}{\sqrt{1^2+(-2)^2}}=\frac{1}{\sqrt{5}}(\hat{i}-2\hat{j})$

$\therefore$ The required vector of magnitude 7 in the direction of $\vec{a}=7\cdot\hat{a}=\frac{7}{\sqrt{5}}(\hat{i}-2\hat{j}).$

OR

Projection of $\vec{a}$ on $\vec{b}=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$=\frac{(7\hat{i}+\hat{j}-4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{\sqrt{(2)^2+(6)^2+(3)^2}}=\frac{14+6-12}{7}=\frac{8}{7}$

4. Given, $\vec{a}=\vec{b}$

$\Rightarrow \quad x\hat{i}+2\hat{j}-z\hat{k}=3\hat{i}-y\hat{j}+\hat{k}$

$\therefore \quad x=3, y=-2, z=-1$

Hence, the value of $x+y+z=0$

5. Vector $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$=(-5\hat{i}+7\hat{j})-(2\hat{i}+\hat{j})=-7\hat{i}+6\hat{j}$

So, its scalar components are $(-7, 6)$.

OR

Given, $|\vec{a}|=3, |\vec{b}|=\frac{2}{3}, |\vec{a}\times\vec{b}|=1$

$\Rightarrow |\vec{a}||\vec{b}|\sin\theta=1 \Rightarrow 3.\frac{2}{3}\sin\theta=1$

$\Rightarrow \sin\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{6}$

6. The given vector is $\vec{a}=3\hat{i}-2\hat{j}+6\hat{k}$

$\Rightarrow |\vec{a}|=\sqrt{3^2+(-2)^2+6^2}=7$

$\therefore$ A unit vector in the direction of vector $\vec{a}$ is

$\hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{1}{7}(3\hat{i}-2\hat{j}+6\hat{k})$

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7. We have, $2\hat{i}-3\hat{j}+4\hat{k}$ and $a\hat{i}+6\hat{j}-8\hat{k}$

Two vectors are collinear if and only if,

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}\Rightarrow\frac{2}{a}=\frac{-3}{6}=\frac{4}{-8}=\frac{-1}{2}\Rightarrow\frac{2}{a}=\frac{-1}{2}\Rightarrow a=-4$$

OR

Here, $|\vec{a}|=\sqrt{3},|\vec{b}|=2$ and angle between $\vec{a}$ and $\vec{b}$ is $60°$.

Now $\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos60°=\sqrt{3}\times2\times\frac{1}{2}=\sqrt{3}$.

8. $(\hat{k}\times\hat{j})\cdot\hat{i}+\hat{j}\cdot\hat{k}=-\hat{i}\cdot\hat{i}+\hat{j}\cdot\hat{k}=-1+0=-1$

9. If $\left(\frac{1}{2},\frac{1}{3},n\right)$ are the dc's of a line, then using the relation $l^2+m^2+n^2=1$, we get

$$\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2+n^2=1\Rightarrow n=\frac{\sqrt{23}}{6}$$

OR

Since, C is the mid point of $A(2,-1)$ and $B(-4,3)$.

$\therefore$ Coordinates of C is $\left(\frac{2-4}{2},\frac{-1+3}{2}\right)=(-1,1)$

$\therefore$ $\overrightarrow{OC}=-\hat{i}+\hat{j}$

10. $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b}=|\vec{a}|^2-|\vec{b}|^2$

11. The given vectors are at right angles if their dot product vanishes.

$\therefore$ $(2\hat{i}-4\hat{j}+\hat{k})\cdot(4\hat{i}-8\hat{j}+\lambda\hat{k})=0$

$\Rightarrow$ $8+32+\lambda=0\Rightarrow\lambda=-40$

12. Area of $\Delta ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}|2\hat{i}-4\hat{j}+4\hat{k}|$

$=\frac{1}{2}\left[\sqrt{(2)^2+(-4)^2+(4)^2}\right]=3$ sq. units

13. For parallel vectors $\vec{a}\times\vec{b}=\vec{0}$

$$\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&1&3\\4&-\lambda&6\end{vmatrix}=0\Rightarrow\hat{i}(6+3\lambda)-\hat{j}(0)+\hat{k}(-2\lambda-4)=0$$

$\therefore$ $6+3\lambda=0\Rightarrow\lambda=-2$

14. We know that $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2$

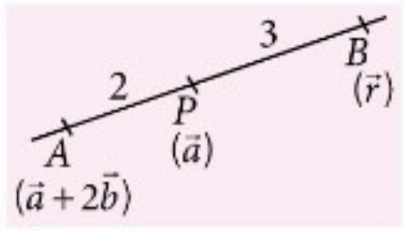
$\Rightarrow$ $(25)^2+|\vec{a}\cdot\vec{b}|^2=(5)^2\times(13)^2\Rightarrow\vec{a}\cdot\vec{b}=\sqrt{3600}=\pm60$

15. Let position vector of B be $\vec{r}$.

$\therefore$ $\vec{a}=\frac{2\vec{r}+3(\vec{a}+2\vec{b})}{2+3}$

$\Rightarrow$ $5\vec{a}=2\vec{r}+3\vec{a}+6\vec{b}$

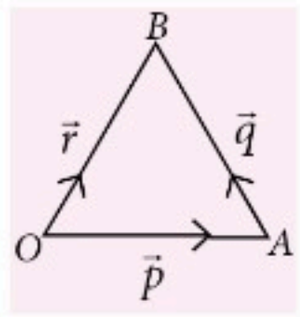
$\Rightarrow$ $2\vec{r}=2\vec{a}-6\vec{b}\Rightarrow\vec{r}=\vec{a}-3\vec{b}$



16. The projection of the vector $(\hat{i}+\hat{j}+\hat{k})$ along the vector $\hat{j}$ is $(\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{\hat{j}}{\sqrt{0^2+1^2+0^2}}\right)=1$

17. **(i) (a) :** Let OAB be a triangle such that

$OA=\vec{p}$, $AB=\vec{q}$, $OB=\vec{r}$

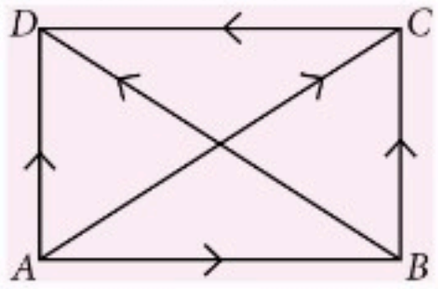


Now, $\vec{p}+\vec{q}=\overrightarrow{OA}+\overrightarrow{AB}$

$=\overrightarrow{OB}=\vec{r}$

(ii) (c) : From triangle law of vector addition,

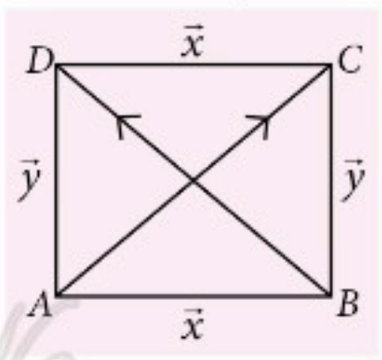
$\overrightarrow{AC}+\overrightarrow{BD}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{BC}+\overrightarrow{CD}$



$=\overrightarrow{AB}+2\overrightarrow{BC}+\overrightarrow{CD}$

$=\overrightarrow{AB}+2\overrightarrow{BC}-\overrightarrow{AB}=2\overrightarrow{BC}$ $[\because\ \overrightarrow{AB}=-\overrightarrow{CD}]$

(iii) (b) : In ΔABC, $\overrightarrow{AC}=\vec{x}+\vec{y}$ ...(i)



and in ΔABD, $\vec{y}=\vec{x}+\overrightarrow{BD}$ ...(ii)

[By triangle law of addition]

Adding (i) and (ii) we have

$\overrightarrow{AC}+\vec{y}=2\vec{x}+\vec{y}+\overrightarrow{BD}$

$\Rightarrow$ $\overrightarrow{AC}-\overrightarrow{BD}=2\vec{x}$

(iv) (d) : In ΔABC, $\overrightarrow{BA}+\overrightarrow{AC}=\overrightarrow{BC}$ ...(i)

[By triangle law]

In ΔBCD, $\overrightarrow{BC}+\overrightarrow{CD}=\overrightarrow{BD}$ ...(ii)

From (i) and (ii) $\overrightarrow{BA}+\overrightarrow{AC}=\overrightarrow{BD}-\overrightarrow{CD}$

$\Rightarrow$ $\overrightarrow{BA}+\overrightarrow{CD}=\overrightarrow{BD}-\overrightarrow{AC}=\overrightarrow{BD}+\overrightarrow{CA}$

(v) (b) : Since S is the mid point of QR

So, $\overrightarrow{QS}=\overrightarrow{SR}$

Now $\overrightarrow{PQ}+\overrightarrow{PR}=(\overrightarrow{PS}+\overrightarrow{SQ})+(\overrightarrow{PS}+\overrightarrow{SR})$ (By triangle law)

$=2\overrightarrow{PS}+\overrightarrow{SQ}+\overrightarrow{SR}=2\overrightarrow{PS}$ $[\because\ \overrightarrow{SQ}=-\overrightarrow{QS}]$

18. **(i) (b) :** Here (2, 11) are the coordinates of A.

$\therefore$ PV of $A=2\hat{i}+11\hat{j}$

(ii) (c) : Here (7, 7) are the coordinates of B.

$\therefore$ PV of $B=7\hat{i}+7\hat{j}$

(iii) (b) : Here PV of $A=2\hat{i}+11\hat{j}$ and PV of $C=2\hat{i}+2\hat{j}$

So, $x_1=2, x_2=2, y_1=11, y_2=2$

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$\therefore \quad \overrightarrow{AC} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$

$= (2-2)\hat{i} + (2-11)\hat{j} = -9\hat{j}$

(iv) (a) : Here $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\therefore \quad |\vec{A}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$

$\therefore \quad \hat{A} = \dfrac{\vec{A}}{|\vec{A}|} = \dfrac{\hat{i}+2\hat{j}+3\hat{j}}{\sqrt{14}} = \dfrac{1}{\sqrt{14}}\hat{i} + \dfrac{2}{\sqrt{14}}\hat{j} + \dfrac{3}{\sqrt{14}}\hat{k}$

(v) (d) : We have $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 3\hat{i} + 2\hat{j}$

$\therefore \quad |\vec{A}| = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$

and $|\vec{B}| = \sqrt{3^2 + 2^2} = \sqrt{9+4} = \sqrt{13}$

Thus, $|\vec{A}| = |\vec{B}| = \sqrt{13}$

19. Let $\vec{c} = \vec{a} + \lambda\vec{b} = (1+\lambda)\hat{i} + (1-\lambda)\hat{j} + (\lambda-1)\hat{k}$

Also, $\vec{c} \cdot \vec{a} = 0$

$\Rightarrow \quad [(1+\lambda)\hat{i} + (1-\lambda)\hat{j} + (\lambda-1)\hat{k}] \cdot [\hat{i} + \hat{j} - \hat{k}] = 0$

$\Rightarrow \quad 1 + \lambda + 1 - \lambda - \lambda + 1 = 0 \Rightarrow \lambda = 3$

$\therefore \quad \vec{c} = 4\hat{i} - 2\hat{j} + 2\hat{k} \quad \Rightarrow \quad \hat{c} = \pm\dfrac{2\hat{i} - \hat{j} + \hat{k}}{\sqrt{6}}$

20. We have, $2\left|\hat{i} + x\hat{j} + 3\hat{k}\right| = \left|4\hat{i} + (4x-2)\hat{j} + 2\hat{k}\right|$

$\Rightarrow \quad 2\sqrt{1 + x^2 + 9} = \sqrt{4^2 + (4x-2)^2 + 2^2}$

$\Rightarrow \quad 4(x^2 + 10) = 16 + 16x^2 + 4 - 16x + 4$

$\Rightarrow \quad 12x^2 - 16x - 16 = 0 \Rightarrow 3x^2 - 4x - 4 = 0$

$\Rightarrow \quad (3x + 2)(x - 2) = 0$

$\Rightarrow \quad x = 2, -\dfrac{2}{3}$

OR

Given that $|\vec{a}| = 2\sqrt{2}, |\vec{b}| = 3$

One diagonal is $5\vec{a} + 2\vec{b} + \vec{a} - 3\vec{b} = 6\vec{a} - \vec{b}$

Its length = $|6\vec{a} - \vec{b}|$

$= \sqrt{36\vec{a}^2 + \vec{b}^2 - 2\times 6|\vec{a}||\vec{b}|\cos 45^\circ}$

$= \sqrt{36\times 8 + 9 - 12\times 2\sqrt{2}\times 3\times\dfrac{1}{\sqrt{2}}} = \sqrt{288 + 9 - 12\times 6} = \sqrt{225} = 15$

$\therefore \quad$ Other diagonal is $5\vec{a} + 2\vec{b} - \vec{a} + 3\vec{b} = 4\vec{a} + 5\vec{b}$

Its length = $\sqrt{16\times 8 + 25\times 9 + 40\times 6} = \sqrt{593}$

21. Let $\vec{A} = \vec{a} + \vec{b} = (\hat{i} + \hat{j} + \hat{k}) + (\hat{i} + 3\hat{j} + 5\hat{k}) = 2\hat{i} + 4\hat{j} + 6\hat{k}$

and $\vec{B} = \vec{b} + \vec{c} = (\hat{i} + 3\hat{j} + 5\hat{k}) + (7\hat{i} + 9\hat{j} + 11\hat{k}) = 8\hat{i} + 12\hat{j} + 16\hat{k}$

If $\vec{A}$ and $\vec{B}$ are diagonals, then area of parallelogram

$= \dfrac{1}{2}|\vec{A} \times \vec{B}| = \dfrac{1}{2}\begin{Vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & 6 \\ 8 & 12 & 16 \end{Vmatrix}$

$= \dfrac{1}{2}\left|-8\hat{i} + 16\hat{j} - 8\hat{k}\right| = \sqrt{(-4)^2 + (8)^2 + (-4)^2} = 4\sqrt{6}$ sq.units

22. $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = -\hat{i} - 2\hat{j} + 3\hat{k}$

$\therefore \quad \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ -1 & -2 & 3 \end{vmatrix} = 5\hat{i} - 7\hat{j} - 3\hat{k}$

23. Position vector which divides the line segment joining points with position vectors $3\vec{a} + \vec{b}$ and $\vec{a} - 3\vec{b}$ in the ratio 2 : 1 externally is given by

$\dfrac{2(\vec{a} - 3\vec{b}) - 1(3\vec{a} + \vec{b})}{2-1} = \dfrac{2\vec{a} - 6\vec{b} - 3\vec{a} - \vec{b}}{1} = -\vec{a} - 7\vec{b}$

OR

Let $A(2\hat{i} - \hat{j} + \hat{k})$, $B(3\hat{i} + 7\hat{j} + \hat{k})$ and

$C(5\hat{i} + 6\hat{j} + 2\hat{k})$

Then, $\overrightarrow{AB} = (3-2)\hat{i} + (7+1)\hat{j} + (1-1)\hat{k} = \hat{i} + 8\hat{j}$

$\overrightarrow{AC} = (5-2)\hat{i} + (6+1)\hat{j} + (2-1)\hat{k} = 3\hat{i} + 7\hat{j} + \hat{k}$

$\overrightarrow{BC} = (5-3)\hat{i} + (6-7)\hat{j} + (2-1)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$

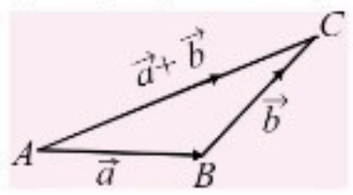
Now, angle between $\overrightarrow{AC}$ and $\overrightarrow{BC}$ is given by

$\cos\theta = \dfrac{\overrightarrow{AC} \cdot \overrightarrow{BC}}{|\overrightarrow{AC}||\overrightarrow{BC}|} = \dfrac{6 - 7 + 1}{\sqrt{9 + 49 + 1}\sqrt{4 + 1 + 1}}$

$\Rightarrow \quad \cos\theta = 0 \Rightarrow AC \perp BC$

So, A, B, C are the vertices of right angled triangle.

24. The inequality holds trivially in case either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$. So, let $|\vec{a}| \neq \vec{0} \neq |\vec{b}|$. Then,



$|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$

$= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$

$= |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$ (scalar product is commutative)

$\leq |\vec{a}|^2 + 2|\vec{a} \cdot \vec{b}| + |\vec{b}|^2$ (Since $x \leq |x|$ $x \in R$)

$\leq |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2 = (|\vec{a}| + |\vec{b}|)^2$

Hence, $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

25. We have, $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \cdot \vec{c} = 0$

$\Rightarrow \quad \vec{a} \perp \vec{b}$ and $\vec{a} \perp \vec{c}$

$\Rightarrow \quad \vec{a}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$

$\Rightarrow \quad \vec{a}$ is parallel to $\vec{b} \times \vec{c}$

$\Rightarrow \quad \vec{a} = \lambda(\vec{b} \times \vec{c})$ for some scalar λ.

$\Rightarrow \quad |\vec{a}| = |\lambda||\vec{b} \times \vec{c}| \Rightarrow |\vec{a}| = |\lambda||\vec{b}||\vec{c}|\sin\dfrac{\pi}{6}$

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$\Rightarrow \quad 1 = \dfrac{|\lambda|}{2} \qquad \left[\because |\vec{a}| = |\vec{b}| = |\vec{c}| = 1\right]$

$\Rightarrow \quad |\lambda| = 2 \Rightarrow \lambda = \pm 2$

$\therefore \quad \vec{a} = \lambda(\vec{b} \times \vec{c}) \Rightarrow \vec{a} = \pm 2(\vec{b} \times \vec{c})$

26. Let $\vec{\beta}_1 = \lambda\vec{\alpha}$, λ is a scalar, *i.e.*, $\vec{\beta}_1 = 3\lambda\hat{i} - \lambda\hat{j}$

Now, $\vec{\beta}_2 = \vec{\beta} - \vec{\beta}_1 = (2 - 3\lambda)\hat{i} + (1 + \lambda)\hat{j} - 3\hat{k}$

Now, since $\vec{\beta}_2$ is to be perpendicular to $\vec{\alpha}$, we should have $\vec{\alpha} \cdot \vec{\beta}_2 = 0$ *i.e.*, $3(2 - 3\lambda) - (1 + \lambda) = 0$ or $\lambda = \dfrac{1}{2}$

Therefore, $\vec{\beta}_1 = \dfrac{3}{2}\hat{i} - \dfrac{1}{2}\hat{j}$ and $\vec{\beta}_2 = \dfrac{1}{2}\hat{i} + \dfrac{3}{2}\hat{j} - 3\hat{k}$

27. We know that, two non zero vectors are perpendicular if their scalar product is zero.

Here,

$\vec{a} + \vec{b} = (5\hat{i} - \hat{j} - 3\hat{k}) + (\hat{i} + 3\hat{j} - 5\hat{k}) = 6\hat{i} + 2\hat{j} - 8\hat{k}$

$\vec{a} - \vec{b} = (5\hat{i} - \hat{j} - 3\hat{k}) - (\hat{i} + 3\hat{j} - 5\hat{k}) = 4\hat{i} - 4\hat{j} + 2\hat{k}$

So, $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = (6\hat{i} + 2\hat{j} - 8\hat{k}) \cdot (4\hat{i} - 4\hat{j} + 2\hat{k})$

$= 24 - 8 - 16 = 0$

Hence, $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular vectors.

28. $\vec{p} = -3\hat{i} + 4\hat{j} - 7\hat{k}$ and $\vec{q} = 6\hat{i} + 2\hat{j} - 3\hat{k}$

$$\therefore \quad \vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 4 & -7 \\ 6 & 2 & -3 \end{vmatrix}$$

$= (-12 + 14)\hat{i} - (9 + 42)\hat{j} + (-6 - 24)\hat{k}$

$\Rightarrow \quad \vec{p} \times \vec{q} = 2\hat{i} - 51\hat{j} - 30\hat{k}$

Now, $\vec{p} \cdot (\vec{p} \times \vec{q}) = (-3\hat{i} + 4\hat{j} - 7\hat{k}) \cdot (2\hat{i} - 51\hat{j} - 30\hat{k}) = 0$

Hence, $\vec{p}$ and $\vec{p} \times \vec{q}$ are perpendicular.

OR

We have, $\vec{a} = -\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$

Now, $\vec{a} + \vec{b} = (-\hat{i} + \hat{j} + \hat{k}) + (2\hat{i} + \hat{j} - 3\hat{k}) = \hat{i} + 2\hat{j} - 2\hat{k}$

Now, a unit vector in the direction of $\vec{a} + \vec{b}$ is

$$\frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \frac{\hat{i} + 2\hat{j} - 2\hat{k}}{\sqrt{1^2 + 2^2 + (-2)^2}} = \frac{\hat{i} + 2\hat{j} - 2\hat{k}}{\sqrt{1 + 4 + 4}} = \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}$$

29. **(i)** If vectors $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a} \cdot \vec{b} = 0$

$\Rightarrow \quad (3\hat{i} + 2\hat{j} + 9\hat{k}) \cdot (\hat{i} + p\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \quad 3 + 2p + 27 = 0 \Rightarrow p = -15$

(ii) Given vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ will be parallel iff

$\dfrac{3}{1} = \dfrac{2}{p} = \dfrac{9}{3} \Rightarrow 3 = \dfrac{2}{p} \Rightarrow p = \dfrac{2}{3}$

30. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the position vectors of points A, B and C respectively.

Then, $\vec{a} = 3i - j + 2k$, $b = i - j - 3k$

and $\vec{c} = 4\hat{i} - 3\hat{j} + \hat{k}$

Now, $\overrightarrow{AB}$ = Position vector of B – Position vector of A

$\Rightarrow \quad \overrightarrow{AB} = \vec{b} - \vec{a}$

$\Rightarrow \quad \overrightarrow{AB} = (\hat{i} - \hat{j} - 3\hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k}) = -2\hat{i} + 0\hat{j} - 5\hat{k}$

$\overrightarrow{AC}$ = Position vector of C – Position vector of A

$\Rightarrow \quad \overrightarrow{AC} = \vec{c} - \vec{a}$

$\Rightarrow \quad \overrightarrow{AC} = (4\hat{i} - 3\hat{j} + \hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k}) = \hat{i} - 2\hat{j} - \hat{k}$

$$\therefore \quad \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix} = (0 - 10)\hat{i} - (2 + 5)\hat{j} + (4 - 0)\hat{k}$$

$\Rightarrow \quad \overrightarrow{AB} \times \overrightarrow{AC} = -10\hat{i} - 7\hat{j} + 4\hat{k}$

$\Rightarrow \quad |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-10)^2 + (-7)^2 + 4^2} = \sqrt{165}$

So, area of $\Delta ABC = \dfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \dfrac{1}{2}\sqrt{165}$ sq. units

31. The vector $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to the vector and $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\therefore \quad$ Required vector $= \dfrac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|}$

Let $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$

Now, $\overrightarrow{AB} = (\hat{i} - \hat{j} - 3\hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k}) = -2\hat{i} - 5\hat{k}$

and $\overrightarrow{AC} = (4\hat{i} - 3\hat{j} + \hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k}) = \hat{i} - 2\hat{j} - \hat{k}$

$$\therefore \quad \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix}$$

$= (0 - 10)\hat{i} - (2 + 5)\hat{j} + (4 - 0)\hat{k} = -10\hat{i} - 7\hat{j} + 4\hat{k}$

$\Rightarrow \quad |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-10)^2 + (-7)^2 + (4)^2} = \sqrt{165}$

Hence, required vector

$= \dfrac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|} = \dfrac{1}{\sqrt{165}}(-10\hat{i} - 7\hat{j} + 4\hat{k})$

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OR

We have $\vec{a} = \hat{i} + \hat{j},\ \vec{b} = \hat{j} - 3\hat{k}$ and $\vec{c} = \hat{i} + 4\hat{k}$

Now, $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -3 \\ 1 & 0 & 4 \end{vmatrix} = (4-0)\hat{i} - (0+3)\hat{j} + (0-1)\hat{k}$

$\Rightarrow \quad \vec{b} \times \vec{c} = 4\hat{i} - 3\hat{j} - \hat{k}$ and

$\therefore \quad \text{L.H.S} = \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 4 & -3 & -1 \end{vmatrix}$

$= (-1-0)\hat{i} - (-1-0)\hat{j} + (-3-4)\hat{k}$

$\Rightarrow \quad \vec{a} \times (\vec{b} \times \vec{c}) = -\hat{i} + \hat{j} - 7\hat{k}$

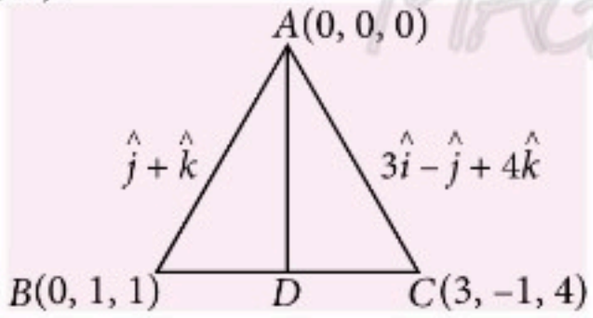
R.H.S. $= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$= [(\hat{i} + \hat{j}) \cdot (\hat{i} + 4\hat{k})](\hat{j} - 3\hat{k}) - [(\hat{i} + \hat{j}) \cdot (\hat{j} - 3\hat{k})](\hat{i} + 4\hat{k})$

$= (\hat{j} - 3\hat{k}) - (\hat{i} + 4\hat{k}) = -\hat{i} + \hat{j} - 7\hat{k} = \text{L.H.S.}$

32. Take A to be as origin (0, 0, 0).

$\therefore$ Coordinates of B are (0, 1, 1) and coordinates of C are (3, –1, 4).



Let D be the mid point of BC and AD is a median of ΔABC.

$\therefore$ Coordinates of D are $\left(\frac{3}{2}, 0, \frac{5}{2}\right)$

So, length of $AD = \sqrt{\left(\frac{3}{2} - 0\right)^2 + (0)^2 + \left(\frac{5}{2} - 0\right)^2}$

$= \sqrt{\frac{9}{4} + \frac{25}{4}} = \frac{\sqrt{34}}{2}$ units

33. *Refer to Answer 99, page no. 245 of MTG CBSE Champion Mathematics, Class 12.*

34. $\vec{a} \times \vec{b}$ is defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$ where, θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system.

We know that, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ ...(i)

and $|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}| \cos\theta$ ... (ii)

Dividing (i) by (ii), we get

$\frac{|\vec{a} \times \vec{b}|}{|\vec{a} \cdot \vec{b}|} = \frac{|\vec{a}||\vec{b}| \sin\theta}{|\vec{a}||\vec{b}| \cos\theta} \Rightarrow |\vec{a} \times \vec{b}| = |\vec{a} \cdot \vec{b}| \tan\theta$

OR

We have, $\vec{a} = 2\hat{i} - \hat{j} + \hat{k},\ \vec{b} = \hat{i} + 3\hat{j} - \hat{k},\ \vec{c} = -2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{d} = 3\hat{i} + 2\hat{j} + 5\hat{k}$.

Now, $\vec{d} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$

$\therefore \quad 3\hat{i} + 2\hat{j} + 5\hat{k} = \alpha(2\hat{i} - \hat{j} + \hat{k}) + \beta(\hat{i} + 3\hat{j} - \hat{k}) + \gamma(-2\hat{i} + \hat{j} - 3\hat{k})$

$\Rightarrow \quad 3\hat{i} + 2\hat{j} + 5\hat{k} = (2\alpha + \beta - 2\gamma)\hat{i} + (-\alpha + 3\beta + \gamma)\hat{j} + (\alpha - \beta - 3\gamma)\hat{k}$

$\Rightarrow \quad 2\alpha + \beta - 2\gamma = 3,\ -\alpha + 3\beta + \gamma = 2,\ \alpha - \beta - 3\gamma = 5$

Solving these equations, we get $\alpha = \frac{-3}{2},\ \beta = 1,\ \gamma = \frac{-5}{2}$

35. Given, $|\vec{a}| = 3,\ |\vec{b}| = 4, |\vec{c}| = 5$ ...(i)

and $\vec{a} \cdot (\vec{b} + \vec{c}) = 0,\ \vec{b} \cdot (\vec{c} + \vec{a}) = 0,\ \vec{c} \cdot (\vec{a} + \vec{b}) = 0$

$\therefore \quad \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} = 0 + 0 + 0 = 0$

$\Rightarrow 2(\vec{a} \cdot \vec{b}) + 2(\vec{b} \cdot \vec{c}) + 2(\vec{c} \cdot \vec{a}) = 0$ ...(ii)

Now $|\vec{a} + \vec{b} + \vec{c}|^2 = (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c})$

$= (\vec{a} + \vec{b} + \vec{c}) \cdot \vec{a} + (\vec{a} + \vec{b} + \vec{c}) \cdot \vec{b} + (\vec{a} + \vec{b} + \vec{c}) \cdot \vec{c}$

$= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{c}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b}) + 2(\vec{b} \cdot \vec{c}) + 2(\vec{c} \cdot \vec{a})$

$= 3^2 + 4^2 + 5^2 + 0$ [Using (i) and (ii)]

$= 50$

$\therefore |\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$.

36. Note that if θ is the angle between AB and CD, then θ is also the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Now, $\overrightarrow{AB}$ = Position vector of B – Position vector of A

$= (2\hat{i} + 5\hat{j}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{i} + 4\hat{j} - \hat{k}$

Therefore, $|\overrightarrow{AB}| = \sqrt{(1)^2 + (4)^2 + (-1)^2} = 3\sqrt{2}$

Similarly, $\overrightarrow{CD} = -2\hat{i} - 8\hat{j} + 2\hat{k}$ and $|\overrightarrow{CD}| = 6\sqrt{2}$

Thus, $\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|}$

$= \frac{1(-2) + 4(-8) + (-1)(2)}{(3\sqrt{2})(6\sqrt{2})} = \frac{-36}{36} = -1$

Since $0 \le \theta \le \pi$, it follows that $\theta = \pi$. This shows that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

OR

Let $\vec{a} = \vec{\alpha} + \vec{\beta}$

where $\vec{\alpha}$ is parallel to $\vec{b}$ and $\vec{\beta}$ is perpendicular to $\vec{b}$

$\therefore \quad \vec{\alpha} = \lambda\vec{b}$, λ is a scalar

i.e., $\vec{\alpha} = \lambda(2\hat{i} + 4\hat{j} - 2\hat{k}) = 2\lambda\hat{i} + 4\lambda\hat{j} - 2\lambda\hat{k}$

and $\vec{\beta} = \vec{a} - \vec{\alpha} = (2\hat{i} - \hat{j} + 3\hat{k}) - (2\lambda\hat{i} + 4\lambda\hat{j} - 2\lambda\hat{k})$

$\Rightarrow \quad \vec{\beta} = (2 - 2\lambda)\hat{i} - (1 + 4\lambda)\hat{j} + (3 + 2\lambda)\hat{k}$

$\because$ $\vec{\beta}$ is perpendicular to $\vec{b}$

$\therefore \quad \vec{\beta} \cdot \vec{b} = 0$

$$\Rightarrow \left((2-2\lambda)\hat{i} - (1+4\lambda)\hat{j} + (3+2\lambda)\hat{k}\right) \cdot (2\hat{i} + 4\hat{j} - 2\hat{k}) = 0$$

$\Rightarrow \quad 2(2 - 2\lambda) - 4(1 + 4\lambda) - 2(3 + 2\lambda) = 0$

$\Rightarrow \quad 4 - 4\lambda - 4 - 16\lambda - 6 - 4\lambda = 0$

$\Rightarrow \quad 24\lambda = -6 \Rightarrow \lambda = \dfrac{-1}{4}$

$\therefore \quad \vec{\alpha} = \dfrac{-1}{4}(2\hat{i} + 4\hat{j} - 2\hat{k}) = \dfrac{-1}{2}\hat{i} - \hat{j} + \dfrac{1}{2}\hat{k}$ and

$$\vec{\beta} = \left(2 + \frac{1}{2}\right)\hat{i} - (1-1)\hat{j} + \left(3 - \frac{1}{2}\right)\hat{k} = \frac{5}{2}\hat{i} + \frac{5}{2}\hat{k} = \frac{5}{2}(\hat{i} + \hat{k})$$

37. Since $\vec{a}$ and $\vec{b}$ are unit vectors $\therefore \quad |\vec{a}| = |\vec{b}| = 1$

We have,

$$|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$$

$$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2|\vec{a}||\vec{b}|\cos\theta + |\vec{b}|^2$$

$$\left[\because \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta\right]$$

$$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = 2 + 2\cos\theta \quad \left[\because |\vec{a}| = |\vec{b}| = 1\right]$$

$$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = 2(1 + \cos\theta) = 2\left(2\cos^2\frac{\theta}{2}\right)$$

$$\left[\because 1 + \cos\theta = 2\cos^2\frac{\theta}{2}\right]$$

$$\Rightarrow \quad 4\cos^2\frac{\theta}{2} = |\vec{a} + \vec{b}|^2 \Rightarrow \quad \cos^2\frac{\theta}{2} = \frac{1}{4}|\vec{a} + \vec{b}|^2$$

$$\Rightarrow \quad \cos\frac{\theta}{2} = \frac{1}{2}|\vec{a} + \vec{b}|$$

OR

We are given, $\vec{a} = 3\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 6\hat{i} + 5\hat{j} - 2\hat{k}$

A unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$.

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -4 \\ 6 & 5 & -2 \end{vmatrix}$

$= (-2 + 20)\hat{i} - (-6 + 24)\hat{j} + (15 - 6)\hat{k} = 18\hat{i} - 18\hat{j} + 9\hat{k}$

and $|\vec{a} \times \vec{b}| = \sqrt{(18)^2 + (-18)^2 + (9)^2}$

$= \sqrt{324 + 324 + 81} = 27$

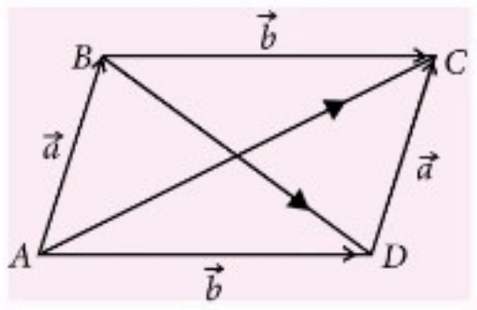
$\therefore \quad \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \dfrac{18\hat{i} - 18\hat{j} + 9\hat{k}}{27} = \dfrac{2}{3}\hat{i} - \dfrac{2}{3}\hat{j} + \dfrac{1}{3}\hat{k}$

Therefore, the vector having magnitude equal to 3 and in the direction of $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ is

$$\frac{3(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|} = 3\left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right) = 2\hat{i} - 2\hat{j} + \hat{k}$$

38. Let $\vec{a} = 2\hat{i} - 4\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + 2\hat{j} + 3\hat{k}$

Then diagonal $\overrightarrow{AC}$ of the parallelogram is



$\vec{p} = \vec{a} + \vec{b}$

$= 2\hat{i} - 4\hat{j} - 5\hat{k} + 2\hat{i} + 2\hat{j} + 3\hat{k}$

$= 4\hat{i} - 2\hat{j} - 2\hat{k}$

Therefore, unit vector parallel to it is

$$\frac{\vec{p}}{|\vec{p}|} = \frac{4\hat{i} - 2\hat{j} - 2\hat{k}}{\sqrt{16 + 4 + 4}} = \frac{2\hat{i} - \hat{j} - \hat{k}}{\sqrt{6}}$$

Now, diagonal $\overrightarrow{BD}$ of the parallelogram is

$\vec{p}' = \vec{b} - \vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k} - 2\hat{i} + 4\hat{j} + 5\hat{k} = 6\hat{j} + 8\hat{k}$

Therefore, unit vector parallel to it is

$$\frac{\vec{p}'}{|\vec{p}'|} = \frac{6\hat{j} + 8\hat{k}}{\sqrt{36 + 64}} = \frac{6\hat{j} + 8\hat{k}}{10} = \frac{3\hat{j} + 4\hat{k}}{5}$$

Now, $\vec{p} \times \vec{p}' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & 6 & 8 \end{vmatrix}$

$= \hat{i}(-16 + 12) - \hat{j}(32 - 0) + \hat{k}(24 - 0) = -4\hat{i} - 32\hat{j} + 24\hat{k}$

$\therefore \quad$ Area of parallelogram $= \dfrac{|\vec{p} \times \vec{p}'|}{2}$

$= \dfrac{\sqrt{16 + 1024 + 576}}{2} = 2\sqrt{101}$ sq. units. 1

OR

Refer to Answer 83, page no. 243 of MTG CBSE Champion Mathematics, Class 12.

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MONTHLY TEST DRIVE

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80 **Series 7 : Definite Integration, Application of Integrals** **Time Taken : 60 Min.**

Only One Option Correct Type

1. If $f(x) = \begin{cases} 3[x] - 5\dfrac{|x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ then $\int_{-3/2}^{2} f(x)dx$, where [.] denotes the greatest integer function, is equal to

(a) -7 (b) -6.5 (c) -6 (d) -5.5

2. The value of $\int_0^1 \dfrac{e^x(1+x)\sin^2(xe^x)dx}{\sin^2(xe^x)+\sin^2(e-e^x x)}$ is

(a) $\dfrac{e}{4}$ (b) $\dfrac{e}{2}$ (c) $2e$ (d) $\dfrac{e}{8}$

3. Area of the ellipse represented by $3x^2 + 4xy + 3y^2 = 1$, is equal to

(a) $\dfrac{\pi}{\sqrt{5}}$ sq. units (b) $\dfrac{\pi}{3\sqrt{5}}$ sq. units

(c) $\dfrac{\pi}{2\sqrt{5}}$ sq. units (d) $\dfrac{\pi}{4\sqrt{5}}$ sq. units

4. If $f''''(x) = k$ in $[0, a]$, then

$$\int_0^a f(x)\,dx - \left\{xf(x) - \frac{x^2}{2!}f'(x) + \frac{x^3}{3!}f''(x)\right\}_0^a \text{ is}$$

(a) $-ka^4/12$ (b) $ka^4/24$

(c) $-ka^4/24$ (d) None of these

5. If $\int_{-1}^{-4} f(x)dx = 4$ and $\int_{2}^{-4} (3 - f(x))\,dx = 7$, then the value of $\int_{-2}^{1} f(-x)dx$, is

(a) 2 (b) -3

(c) 5 (d) None of these

6. If A_n is the area bounded by $y = x$ and $y = x^n$, $n \in N$, then $A_2 \cdot A_3 \ldots A_n =$

(a) $\dfrac{1}{n(n+1)}$ (b) $\dfrac{1}{2^n n(n+1)}$

(c) $\dfrac{1}{2^{n-1} n(n+1)}$ (d) $\dfrac{1}{2^{n-2} n(n+1)}$

One or More Than One Option(s) Correct Type

7. If $I_1 = \int_0^{\pi} \dfrac{x\sin x}{1+\cos^2 x}dx$, $I_2 = \int_0^{\pi} x\sin^4 x\,dx$, then $I_1 : I_2 =$

(a) 3 : 4 (b) 1 : 2 (c) 4 : 3 (d) 2 : 3

8. The integral $\int_0^{\pi} xf(\sin x)\,dx$ is equal to

(a) $\dfrac{\pi}{2}\int_0^{\pi} f(\sin x)dx$ (b) $\dfrac{\pi}{4}\int_0^{\pi} f(\sin x)dx$

(c) $\pi\int_0^{\pi/2} f(\sin x)dx$ (d) $\pi\int_0^{\pi/2} f(\cos x)dx$

9. For which of the following values of m, is the area of the region bounded by the curve $y = x - x^2$ and the line $y = mx$ is equal to 9/2 sq. units?

(a) -4 (b) -2 (c) 2 (d) 4

10. The area enclosed by the curves $x = a\sin^3 t$ and $y = a\cos^3 t$ is equal to

(a) $12a^2\int_0^{\pi/2} \cos^4 t \sin^2 t\,dt$

(b) $12a\int_0^{\pi/2} \cos^2 t \sin^4 t\,dt$

MAGAZINE RING

(c) $2\int_{-a}^{a}\left(a^{2/3}-x^{2/3}\right)^{3/2}dx$

(d) $4\int_{0}^{a}\left(a^{2/3}-x^{2/3}\right)^{3/2}dx$

11. The absolute value of $\int_{10}^{19}\frac{\sin x}{1+x^8}dx$ is

(a) less than 10^{-7} (b) more than 10^{-7}

(c) less than 10^{-6} (d) None of these

12. Area enclosed by parabola $ay = 3(a^2 - x^2)$ and x-axis is 64, then value of a is

(a) 4 (b) 2 (c) −4 (d) −2

13. Consider the function

$$f(x)=\begin{cases}\int_0^x(4+|t-2|)dt, & x>3\\ ax^2+bx, & x\le 3\end{cases}$$

If $f(x)$ is differentiable at $x = 3$, then

(a) $a+b=\frac{83}{18}$ (b) $a+b=\frac{85}{18}$

(c) $ab=\frac{7}{27}$ (d) $\frac{b}{a}=84$

Comprehension Type

Consider $f(x)=\int_0^{\pi}\sin x\cdot f(t)\,dt+\int_0^{\pi/2}\cos x f'(t)dt+2$

14. Range of $\frac{f(x)-2}{\pi}$ is

(a) $\left[-\sqrt{2},\sqrt{2}\right]$ (b) $\left[-\sqrt{5},\sqrt{5}\right]$

(c) $\left[-\sqrt{3},\sqrt{3}\right]$ (d) None of these

15. $\int_{\frac{\pi}{3}}^{\frac{13\pi}{3}} f(x)\,dx$ is equal to

(a) 4π (b) 0 (c) 2 (d) 8π

Matrix Match Type

16. $\lim_{n\to\infty} f(n)$, where $f(n)$ is

Column-I		Column-II	
P.	$\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2-1}}+....+\frac{1}{\sqrt{n^2-(n-1)^2}}\right)$	1.	$\frac{\pi}{4}+\frac{1}{2}\log 2$
Q.	$\left(\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+....+\frac{1}{n}\right)$	2.	1/2
R.	$\frac{1}{n}\left[\sin^2\frac{\pi}{2n}+\sin^2\frac{2\pi}{2n}+....+\sin^2\frac{n\pi}{2n}\right]$	3.	$\pi/6$
S.	$\left(\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-2^2}}+....+\frac{1}{\sqrt{4n^2-n^2}}\right)$	4.	$\pi/2$

	P	Q	R	S
(a)	4	1	2	3
(b)	2	1	3	4
(c)	1	2	3	4
(d)	4	2	1	3

Numerical Value Type

17. A biquadratic function $y = ax^4 + bx^3 + cx^2 + dx + e$ $(a \neq 0)$ touches $y = px + q$ at $x = \alpha, \beta$ $(\alpha < \beta)$. Area of the region bounded by these graphs is $\frac{a(\beta-\alpha)^5}{\lambda}$, then $\frac{\lambda}{5}=$

18. If $I=\int_{-1}^{1}\frac{e^{|x|}}{1+a^x}dx$, then the value of $\frac{9997}{(e-1)}I$ must be

19. If $\int_0^1\frac{dx}{(1+x)(2+x)\sqrt{x(1-x)}}=\frac{\pi A}{\sqrt{6}\left(\sqrt{3}+1\right)(157)}$, then A is equal to

20. If 'a' $(a>0)$ is the value of parameter for each of which the area of the figure bounded by the straight line $y=\frac{a^2-ax}{1+a^4}$ and the parabola $y=\frac{x^2+2ax+3a^2}{1+a^4}$ is the greatest, then the value of a^4 is

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SELF CHECK

No. of questions attempted .....
No. of questions correct .....
Marks scored in percentage .....

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

PRACTICE PROBLEMS JEE MAIN 2021

1. Total number of real values of x, such that $\frac{(12+x)^{1/7}}{x}+\frac{(12+x)^{1/7}}{12}=\frac{64}{3}x^{1/7}$ is equal to

(a) one (b) two
(c) zero (d) none of these

2. If the equations $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ and $x^2 + (a + b)x + 36 = 0$ have a common positive root then ordered pair (a, b) is

(a) $(-6, -8)$ (b) $(-7, -8)$
(c) $(-6, -7)$ (d) $(-7, -9)$

3. If $x_1, x_2, ..., x_n$ are in H.P., then $\sum_{r=1}^{n-1} x_r x_{r+1}$ is equal to

(a) $(n - 1)x_1x_n$ (b) nx_1x_n
(c) $(n + 1)x_1x_n$ (d) none of these

4. If $|z| \le 4$, then the maximum value of $|iz + 3 - 4i|$ is equal to

(a) 2 (b) 4 (c) 3 (d) 9

5. The value of $\sum_{r=0}^{n}(r)^{2n}C_r$ is equal to

(a) $2n(2^{2n-2} + {}^{2n-1}C_{n-1})$
(b) $n(2^{2n-2} + {}^{2n-1}C_n)$
(c) $2n(2^{2n-2} + {}^{2n-1}C_{n+1})$
(d) none of these

6. Total number of ways, in which 22 different books can be given to 5 students, so that two students get 5 books each and all the remaining students get 4 books each, is equal to

(a) $\frac{22!}{3!2!5!(4!)^3}$ (b) $\frac{22!}{(3!)^2\cdot 2!\cdot 5!}$
(c) $\frac{22!}{3!\,2!\,5!4!}$ (d) none of these

7. A team of four students is to be selected from a total of 12 students. Total number of ways in which team can be selected such that two particular students refuse to be together and other two particular students wish to be together only, is equal to

(a) 220 (b) 182
(c) 226 (d) none of these

8. In triangle ABC, the value of $\begin{vmatrix} e^{-i2A} & e^{iC} & e^{iB} \\ e^{iC} & e^{-i2B} & e^{iA} \\ e^{iB} & e^{iA} & e^{-i2C} \end{vmatrix}$ where $i=\sqrt{-1}$, is equal to

(a) -2 (b) -4
(c) -3 (d) none of these

9. Let $\vec{a},\vec{b},\vec{c}$ be three unit vectors such that $\vec{a}+5\vec{b}+3\vec{c}=\vec{0}$ then $\vec{a}\cdot(\vec{b}\times\vec{c})$ is equal to

(a) $\vec{a}\cdot\vec{b}$ (b) $\vec{a}\cdot(\vec{b}+2\vec{c})$
(c) $\vec{b}\cdot(\vec{a}+\vec{c})$ (d) none of these

10. Consider the set of integers {10, 11, 12, ..., 98, 99}. By seeing the number a person A will laugh if the product of the digits of the number is 12. He chooses three numbers one at a time from this set of integers, randomly and with replacement. The probability that he will laugh atleast once, is equal to

(a) $1-\left(\frac{43}{45}\right)^3$ (b) $1-\left(\frac{3}{5}\right)^3$
(c) $1-\left(\frac{4}{25}\right)^3$ (d) none of these

11. The probability that an archer hits the target when it is windy is equal to 2/5. When it is not windy his probability of hitting the target is 7/10. On any shot the probability of gust of wind is 3/10. The probability that there is no gust of wind on the occasion when he missed the target, is equal to

(a) 5/13 (b) 18/39 (c) 7/13 (d) 23/39

12. If A and B are square matrices such that $AB = B$ and $BA = A$, then $A^2 + B^2$ is equal to

(a) $2AB$ (b) $2BA$ (c) $A + B$ (d) AB

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13. Range of $f(x)=\cos^{-1}\left(\frac{1+x^2}{2x}\right)+\sqrt{2-x^2}$ is

(a) $\left\{0,1+\frac{\pi}{2}\right\}$ (b) $\{0, 1+\pi\}$

(c) $\left\{1, 1+\frac{\pi}{2}\right\}$ (d) $\{1, 1+\pi\}$

14. If $[\cos^{-1}x] + [\cot^{-1}x] = 0$, where $[\cdot]$ denotes the greatest integer function, then complete set of values of x is

(a) (cos 1, 1] (b) (cos 1, cot 1)

(c) (cot 1, 1] (d) [0, cot 1)

15. $\lim\limits_{x\to1}\dfrac{(x-1)(x^2-1)(x^3-1)(x^4-1)}{((x-1)(x^2-1))^2}$ is equal to

(a) 2 (b) 3

(c) 6 (d) none of these

16. $f(x) = [\sin x] + [\cos x]$, $x \in [0, 2\pi]$, where $[\cdot]$ denotes the greatest integer function. Total number of points where $f(x)$ is non-differentiable is equal to

(a) 2 (b) 3 (c) 5 (d) 4

17. If $f(x) = x^2 - 4|x|$ and

$g(x)=\begin{cases}\min\{f(t):-6\le t\le x\}, x\in[-6,0]\\ \max\{f(t):0<t\le x\}, x\in(0,6]\end{cases}$, then $g(x)$ has

(a) exactly one point of local minima

(b) exactly one point of local maxima

(c) no point of local maxima but exactly one point of local minima

(d) neither a point of local maxima nor minima

18. $\int \sin x\cdot\cos x\cdot\cos 2x\cdot\cos 4x\cdot\cos 8x\,dx$ is equal to

(a) $\frac{1}{96}\cos 16x+c$ (b) $-\frac{1}{256}\cos 16x+c$

(c) $-\frac{1}{16}\cos 16x+c$ (d) none of these

19. $\int_0^{\frac{3\pi}{2}} \sin\left[\frac{2x}{\pi}\right]dx$ where $[\cdot]$ denotes the greatest integer function, is equal to

(a) $\frac{\pi}{2}(\sin1 + \cos1)$ (b) $\frac{\pi}{2}(\sin1 - \sin2)$

(c) $\frac{\pi}{2}(\sin1 - \cos1)$ (d) $\frac{\pi}{2}(\sin1 + \sin2)$

20. Value of positive real parameter 'a' such that the area of the region bounded by the parabola $y = x - ax^2$, $ay = x^2$ attains its maximum value, is equal to

(a) 1/2 (b) 2

(c) 1 (d) none of these

21. General solution of $x\frac{dy}{dx}+y=y^2x^3\cos x$, is

(a) $\frac{1}{xy}+x\sin x-\cos x=C$ (b) $\frac{1}{xy}-x\sin x+\cos x=C$

(c) $\frac{1}{xy}+x\cos x-\sin x=C$ (d) $\frac{1}{xy}+x\sin x+\cos x=C$

22. A variable line is drawn through the intersection point of the lines $3x + 4y - 12 = 0$ and $x + 2y - 5 = 0$, meeting the co-ordinate axes at the points A and B. Locus of midpoint of segment AB is

(a) $4x + 3y = 4xy$ (b) $3x + 4y = 3xy$

(c) $3x + 4y = 4xy$ (d) none of these

23. If $x - 2y + 4 = 0$ and $2x + y - 5 = 0$ are the sides of an isosceles triangle having area 10 sq. units, then equation of third side is

(a) $x + 3y = 19$ (b) $3x - y + 11 = 0$

(c) $x - 3y = 10$ (d) $3x - y + 15 = 0$

24. The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a variable triangle OAB. Sides OA and OB lie along the x and y-axis respectively, where 'O' is the origin. Locus of midpoint of side AB is $x+y-xy+\lambda\sqrt{x^2+y^2}=0$, then λ is equal to

(a) -1 (b) 1 (c) -2 (d) 2

25. PA and PB are tangents drawn to $y^2 = 4ax$ from arbitrary point P. If the angle between tangents is $\pi/4$, then locus of point P is

(a) $y^2 = x^2 + a^2 + 6ax$ (b) $y^2 = x^2 - a^2 + 6ax$

(c) $y^2 = x^2 + a^2 - 6ax$ (d) $y^2 = x^2 - a^2 - 6ax$

26. The equation of common tangents of the curves $x^2 + 4y^2 = 8$ and $y^2 = 4x$ are

(a) $y - 2x - 4 = 0$, $y + 2x + 4 = 0$

(b) $y - 2x - 2 = 0$, $y + 2x + 2 = 0$

(c) $2y - x - 4 = 0$, $2y + x + 4 = 0$

(d) $2y - x - 2 = 0$, $2y + x + 2 = 0$

27. Length of the latus rectum of the hyperbola $xy = c^2$, is equal to

(a) $2c$ (b) $\sqrt{2}c$ (c) $2\sqrt{2}c$ (d) $4c$

Monthly Test Drive-7 CLASS XI ANSWER KEY

1.	(b)	**2.**	(d)	**3.**	(b)	**4.**	(c)
5.	(a)	**6.**	(b)	**7.**	(a,c,d)	**8.**	(c)
9.	(a,b,c,d)	**10.**	(a,d)	**11.**	(a,c)	**12.**	(a,b,c)
13.	(a,b,c)	**14.**	(c)	**15.**	(c)	**16.**	(a)
17.	(27)	**18.**	(6)	**19.**	(0.25)	**20.**	(2)

28. The equation of the plane passing through the points (2, –1, 0), (3, –4, 5) and parallel to the line $2x = 3y = 4z$ is

(a) $125x - 90y - 79z = 340$

(b) $32x - 21y - 36z = 85$

(c) $73x + 61y - 22z = 85$

(d) $29x - 27y - 22z = 85$

29. Reflection of the line $\frac{x-1}{-1}=\frac{y-2}{3}=\frac{z-4}{1}$ in the plane $x + y + z = 7$ is

(a) $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-4}{1}$ (b) $\frac{x-1}{-3}=\frac{y-2}{-1}=\frac{z-4}{1}$

(c) $\frac{x-1}{-3}=\frac{y-2}{1}=\frac{z-4}{-1}$ (d) $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-4}{-1}$

30. If x, y, z are in A.P., then $\frac{\sin x-\sin z}{\cos z-\cos x}$ is equal to

(a) $\tan y$ (b) $\cot y$ (c) $\sin y$ (d) $\cos y$

31. The minimum value of $27^{\cos x} + 81^{\sin x}$ is equal to

(a) $\frac{2}{3\sqrt{3}}$ (b) $\frac{1}{3\sqrt{3}}$

(c) $\frac{2}{9\sqrt{3}}$ (d) none of these

32. $\sum_{r=1}^{n}\sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right)$ is equal to

(a) $\tan^{-1}(\sqrt{n})-\frac{\pi}{4}$ (b) $\tan^{-1}(\sqrt{n+1})-\frac{\pi}{4}$

(c) $\tan^{-1}(\sqrt{n})$ (d) none of these

33. Equation of the plane such that foot of altitude drawn from (–1, 1, 1) to the plane has the coordinate (3, –2, –1), is

(a) $x + y + z = 0$ (b) $4x - 3y - 2z = 20$

(c) $3x + y - z = 8$ (d) $4x + 3y - z = 7$

34. In a triangle ABC, let $\angle A = \pi/2$. If r is the inradius and R is the circumradius of the triangle ABC, then the value of $r + 2R$ is equal to

(a) $\frac{1}{2}(a + b - c)$ (b) $\frac{1}{2}(b + c - a)$

(c) $\frac{1}{2}(a + c - b)$ (d) $\frac{1}{2}(a + b + c)$

35. If a, b, c are the sides of a triangle ABC, then $\sqrt{a}+\sqrt{b}-\sqrt{c}$ is always

(a) negative (b) positive

(c) non-negative (d) non-positive

36. $\int \tan^{-1}\sqrt{x}\,dx$ is equal to

(a) $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+C$

(b) $x\tan^{-1}\sqrt{x}-\sqrt{x}+C$

(c) $\sqrt{x}-x\tan^{-1}\sqrt{x}+C$

(d) $\sqrt{x}-(x+1)\tan^{-1}\sqrt{x}+C$

37. If the curves $\frac{x^2}{a^2}+\frac{y^2}{4}=1$ and $y^2 = 16x$ intersect at right angle, then

(a) $a = \pm 1$ (b) $a=\pm\sqrt{3}$

(c) $a=\pm\frac{1}{\sqrt{3}}$ (d) $a=\pm\sqrt{2}$

38. If $f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$ is increasing for all values of x, then

(a) $a \in (-\infty, \infty)$ (b) $a \in (-\infty, 0]$

(c) $a \in [0, \infty)$ (d) none of these

39. Domain of $f(x)=\sqrt{[x]-1+x^2}$; where $[\cdot]$ denotes the greatest integer function, is

(a) $(-\infty, -2) \cup [1, \infty)$ (b) $(-\infty,-\sqrt{2})\cup[1, \infty)$

(c) $(-\infty, -3) \cup [1, \infty)$ (d) $(-\infty,-\sqrt{3}]\cup[1, \infty)$

40. $\sum_{r=1}^{n}\frac{r}{(r+1)!}$ is equal to

(a) $1-\frac{1}{(n+1)!}$ (b) $1-\frac{1}{n!}$

(c) $1-\frac{1}{(n-1)!}$ (d) none of these

SOLUTIONS

1. (a) : We have, $\frac{(12+x)^{1/7}}{x}+\frac{(12+x)^{1/7}}{12}=\frac{64}{3}x^{1/7}$

$\Rightarrow (12 + x)(12 + x)^{1/7} = 64 \cdot 4 \cdot x \cdot x^{1/7} = 2^8 \cdot x^{8/7}$

$\Rightarrow 12 + x = x(2^8)^{7/8} \Rightarrow x=\frac{12}{127}$

2. (b) : We have, $x^2 + ax + 12 = 0$ ...(i)

$x^2 + bx + 15 = 0$ ...(ii)

$x^2 + (a + b)x + 36 = 0$ ...(iii)

Adding (i) & (ii) we get

$2x^2 + (a + b)x + 27 = 0$ ...(iv)

Now, subtracting (iii) from (iv), we get

$x^2 - 9 = 0 \Rightarrow x = 3, -3$

Thus, common positive root is 3.

$\therefore\ 9 + 3a + 12 = 0 \Rightarrow a = -7$

and $9 + 3b + 15 = 0 \Rightarrow b = -8$
Thus ordered pair (a, b) is $(-7, -8)$.

3. (a) : Clearly, $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}$ will be in A.P.

$$\Rightarrow \frac{1}{x_2} - \frac{1}{x_1} = \frac{1}{x_3} - \frac{1}{x_2} = \ldots = \frac{1}{x_{r+1}} - \frac{1}{x_r} = \ldots$$

$$= \frac{1}{x_n} - \frac{1}{x_{n-1}} = \lambda \text{ (say)}$$

$$\Rightarrow \sum_{r=1}^{n-1} x_r\, x_{r+1} = \frac{1}{\lambda}\sum_{r=1}^{n-1}(x_r - x_{r+1}) = \frac{1}{\lambda}(x_1 - x_n)$$

Consider, $\frac{1}{x_n} = \frac{1}{x_1} + (n-1)\lambda \Rightarrow \frac{x_1 - x_n}{x_1 x_n} = (n-1)\lambda$

$$\therefore \sum_{r=1}^{n-1} x_r x_{r+1} = (n-1)x_1 x_n$$

4. (d) : $|iz + 3 - 4i| \le |iz| + |3 - 4i| = |z| + 5 \le 9$

5. (d) : We have, $\sum_{r=0}^{n} r \cdot {}^{2n}C_r = \sum_{r=0}^{n} 2n \cdot {}^{2n-1}C_{r-1}$

$= 2n({}^{2n-1}C_0 + {}^{2n-1}C_1 + \ldots + {}^{2n-1}C_{n-1})$

$= 2n(\frac{1}{2}(2^{2n-1})) = 2n(2^{2n-2})$

6. (a) : It is same as dividing 22 books in 5 packets such that two packets have 5 books each and remaining three have four books each and then distributing these packets. Thus, total ways of distributing these books

$$= \frac{22!}{(5!)^2 \cdot (4!)^3 \cdot 2!3!}.(5!) = \frac{22!}{2!3!5!(4!)^3}$$

7. (c) : Let S_1 and S_2 refuse to be together and S_3 and S_4 want to be together only.
Total ways when S_3 and S_4 are selected
$= ({}^8C_2 + {}^2C_1 \cdot {}^8C_1) = 44$
Total ways when S_3 and S_4 are not selected
$= ({}^8C_4 + {}^2C_1 \cdot {}^8C_3) = 182$
Thus total ways $= 44 + 182 = 226$.

8. (b) : Given determinant

$$= e^{-i2A}(e^{-i2(B+C)} - e^{i2A}) - e^{iC}(e^{-iC} - e^{i(A+B)}) + e^{iB}(e^{i(A+C)} - e^{-iB})$$

$$= e^{-i2(A+B+C)} - 1 - 1 + e^{i(A+B+C)} + e^{i(A+B+C)} - 1$$

$$= e^{-i2\pi} - 1 - 1 + e^{i\pi} + e^{i\pi} - 1$$

$$= 1 - 1 - 1 - 1 - 1 - 1 = -4$$

9. (d) : Since, $\vec{a} + 5\vec{b} + 3\vec{c} = \vec{0}$

Thus $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

Hence, $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

10. (a) : Person will laugh if the selected number is 43 or 34 or 62 or 26.
Thus, probability that person will not laugh on seeing the number $= \frac{86}{90} = \frac{43}{45}$

Thus, required probability $= 1 - \left(\frac{43}{45}\right)^3$

11. (c) : Let E_1 : There is a gust of wind.
E_2 : There is no gust of wind.
A : archer misses the target.
Now, $P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)$

$$= \frac{3}{10}\cdot\frac{3}{5} + \frac{7}{10}\cdot\frac{3}{10} = \frac{39}{100}$$

$\therefore$ Required probability $= P(E_2/A) = \frac{P(E_2 \cap A)}{P(A)}$

$$= \frac{P(E_2)\cdot P(A/E_2)}{P(A)} = \frac{\frac{7}{10}\cdot\frac{3}{10}}{\frac{39}{100}} = \frac{7}{13}$$

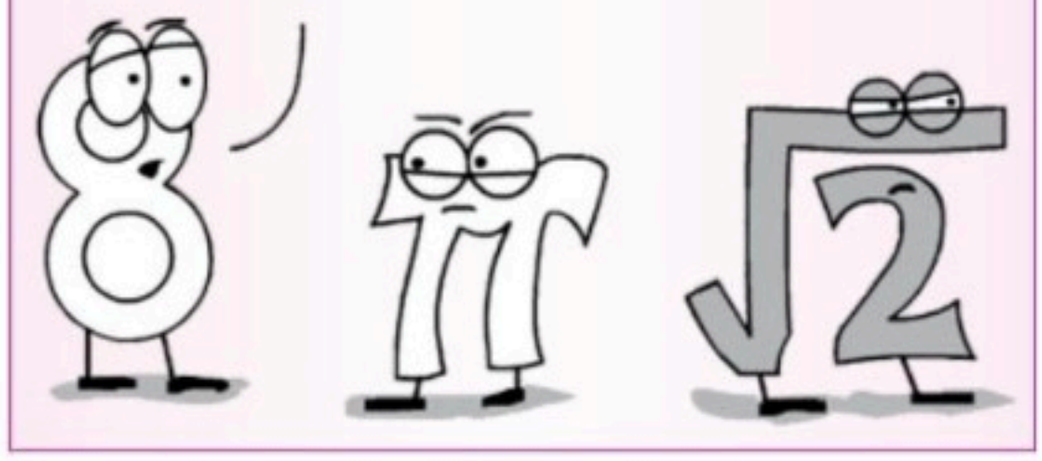
12. (c) : Given, $AB = B$, $BA = A$
Now, $A^2 + B^2 = AA + BB = A(BA) + B(AB)$
$= (AB)A + (BA)B = BA + AB = A + B$

13. (d) : $\cos^{-1}\left(\frac{1+x^2}{2x}\right)$ is defined if $\left|\frac{1+x^2}{2x}\right| \le 1$

$\Rightarrow x \ne 0 \Rightarrow 1 + x^2 - 2|x| \le 0$
$\Rightarrow (|x| - 1)^2 \le 0 \Rightarrow x = 1, -1$
Thus, domain of $f(x)$ is $\{1, -1\}$.
Hence, range is $\{1, 1 + \pi\}$

14. (c) : We have, $[\cos^{-1}x] \ge 0 \ \forall\ x \in [-1, 1]$
and $[\cot^{-1}x] > 0 \ \forall\ x \in R$
Now, $[\cos^{-1}x] + [\cot^{-1}x] = 0 \Rightarrow [\cos^{-1}x] = [\cot^{-1}x] = 0$
If $[\cos^{-1}x] = 0$, then $x \in (\cos 1, 1]$
If $[\cot^{-1}x] = 0$, then $x \in (\cot 1, \infty) \Rightarrow x \in (\cot 1, 1]$

COMIC CAPSULE



15. (c) : We have, $\dfrac{(x-1)(x^2-1)(x^3-1)(x^4-1)}{((x-1)(x^2-1))^2}$

$$=\frac{(x^3-1)(x^4-1)}{(x-1)(x^2-1)} = (x^2+x+1)(x^2+1)$$

$$\therefore \lim_{x\to 1}\frac{(x-1)(x^2-1)(x^3-1)(x^4-1)}{((x-1)(x^2-1))^2}$$

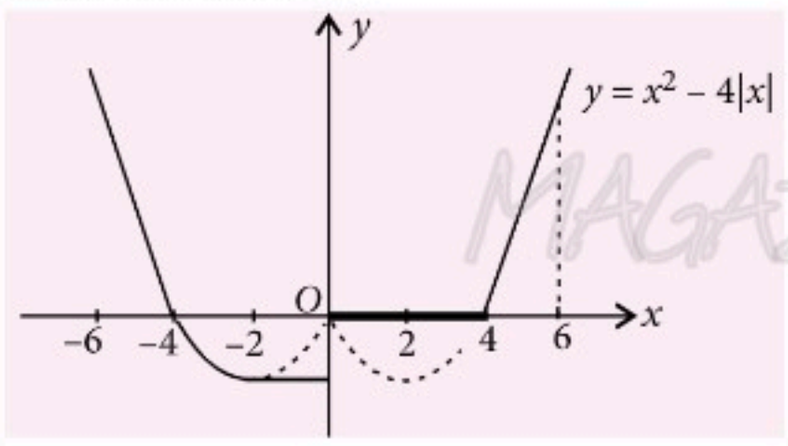
$$=\lim_{x\to 1}(x^2+x+1)(x^2+1)=6$$

16. (c) : $[\sin x]$ is discontinuous at $x = \pi/2, \pi, 2\pi$ and $[\cos x]$ is discontinuous at $x = 0, \pi/2, 3\pi/2, 2\pi$.

Thus, $f(x)$ is discontinuous at $x = 0, \dfrac{\pi}{2}, \pi, 3\pi/2, 2\pi$.

Thus, $f(x)$ is discontinuous and hence non-differentiable at 5 points.

17. (d) : Bold line represents the graph of $y = g(x)$. Clearly, $g(x)$ has neither a point of local maxima nor a point of local minima.



18. (b) : $\int \sin x\cdot\cos x\cdot\cos 2x\cdot\cos 4x\cdot\cos 8x\ dx$

$$=\frac{1}{2}\int \sin 2x\cdot\cos 2x\cdot\cos 4x\cdot\cos 8x\,dx$$

$$=\frac{1}{4}\int \sin 4x\cdot\cos 4x\cdot\cos 8x\,dx$$

$$=\frac{1}{8}\int \sin 8x\cdot\cos 8x\ dx=\frac{1}{16}\int \sin 16x\ dx$$

$$=\frac{-1}{256}\cos 16x+C$$

19. (d) : $\displaystyle\int_0^{3\pi/2}\sin\left[\frac{2x}{\pi}\right]dx$

$$=\int_0^{\pi/2}\sin\left[\frac{2x}{\pi}\right]dx+\int_{\pi/2}^{\pi}\sin\left[\frac{2x}{\pi}\right]dx+\int_{\pi}^{3\pi/2}\sin\left[\frac{2x}{\pi}\right]dx$$

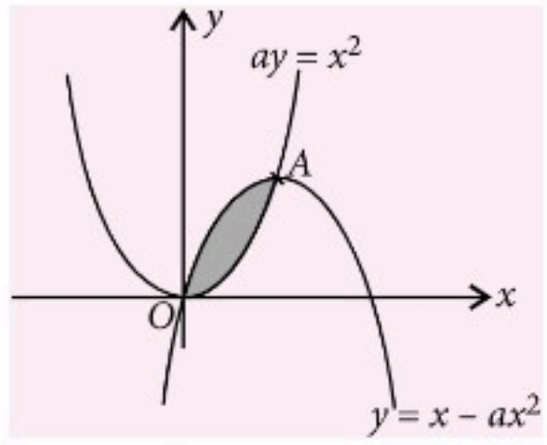
$$=0+\sin 1\int_{\pi/2}^{\pi}dx+\sin 2\int_{\pi}^{3\pi/2}dx \quad =\frac{\pi}{2}(\sin 1+\sin 2)$$

20. (c) : At point A,

$$\frac{x^2}{a}=x-ax^2$$

$$\Rightarrow\quad x=0,\ x=\frac{a}{1+a^2}$$

$\therefore$ Required area,



$$\Delta=\int_0^{a/1+a^2}\left(x-ax^2-\frac{x^2}{a}\right)dx$$

$$\Rightarrow\quad \frac{d\Delta}{da}=\int_0^{a/1+a^2}\left(-x^2+\frac{x^2}{a^2}\right)dx+0\cdot\frac{d}{da}\left(\frac{a}{1+a^2}\right)$$

$$=\left[\frac{(1-a^2)}{3a^2}x^3\right]_0^{a/1+a^2}=\frac{a(1-a^2)}{3(1+a^2)^3}=\frac{a(1-a)(1+a)}{3(1+a^2)^3}$$

$$\Rightarrow\quad \frac{d\Delta}{da}>0\ \ \forall\ \ a\in(0,1)$$

and $\dfrac{d\Delta}{da}<0\ \ \forall\ a\in(1,\infty)$

Thus, $a = 1$ is the point of maxima for Δ.

21. (d) : Given, $x\dfrac{dy}{dx}+y=y^2\cdot x^3\cdot\cos x$

$$\Rightarrow\quad \frac{(xdy+ydx)}{x^2y^2}=x\cdot\cos x dx$$

$$\Rightarrow\quad \frac{d(xy)}{x^2y^2}=x\cdot\cos x dx\quad\Rightarrow\quad -\frac{1}{xy}=x\sin x+\cos x+C$$

22. (c) : Equation of any line drawn through the intersection point of the given lines is,

$$3x+4y-12+\lambda(x+2y-5)=0$$

$$\Rightarrow\quad A\equiv\left(\frac{5\lambda+12}{3+\lambda},0\right),\ B\equiv\left(0,\frac{5\lambda+12}{2\lambda+4}\right)$$

If midpoint of AB is $C(h, k)$, then

$$2h=\frac{5\lambda+12}{3+\lambda},\ 2k=\frac{5\lambda+12}{2\lambda+4}$$

$$\Rightarrow\quad \frac{h}{k}=\frac{2(\lambda+2)}{3+\lambda}\ \Rightarrow\ \lambda=\frac{4k-3h}{h-2k}$$

$$\therefore\quad 2h=\frac{12+\dfrac{5(4k-3h)}{h-2k}}{3+\dfrac{4k-3h}{h-2k}}$$

Thus required locus is, $3x + 4y = 4xy$

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23. (a) : Given lines are mutually perpendicular and intersect at $\left(\frac{6}{5}, \frac{13}{5}\right)$.

Equation of angle bisector of the given lines is

$x - 2y + 4 = \pm(2x + y - 5)$

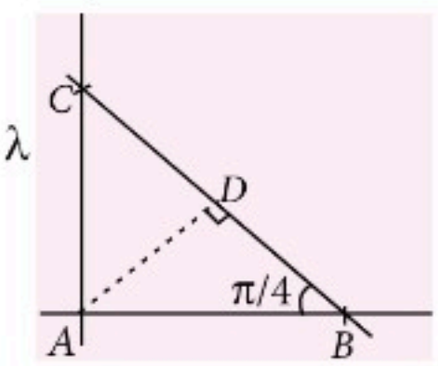
i.e., $x + 3y = 9$, $3x - y = 1$

Side BC will be parallel to these bisectors.

Let $AD = a \Rightarrow AB = a\sqrt{2}$

and, area of $\Delta_{ABC} = \frac{1}{2}(a\sqrt{2})^2$

$\Rightarrow a^2 = 10 \Rightarrow a = \sqrt{10}$



Let equation of BC is $x + 3y = \lambda$

$$\Rightarrow \sqrt{10} = \frac{\left|\frac{6}{5} + \frac{39}{5} - \lambda\right|}{\sqrt{10}}$$

$\Rightarrow \lambda = -1, 19$

Thus equation of BC is

$x + 3y = -1$ or $x + 3y = 19$

If equation of BC is $3x - y = \lambda$

$$\Rightarrow \sqrt{10} = \frac{\left|\frac{18}{5} - \frac{13}{5} - \lambda\right|}{\sqrt{10}} \Rightarrow \lambda = -9, 11$$

Hence, equation of BC is

$3x - y = -9$ or $3x - y = 11$.

24. (b) : Let the equation of side AB be

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{or} \quad bx + ay - ab = 0$$

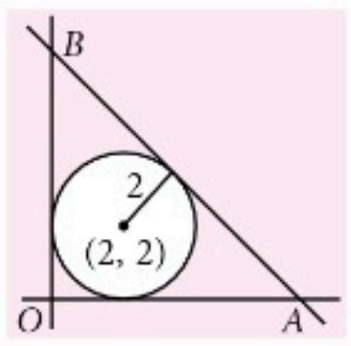
Centre and radius of circle are (2, 2) and 2 units respectively.

Since, AB touches the circles. Therefore,

$$\frac{|2b + 2a - ab|}{\sqrt{a^2 + b^2}} = 2$$

$$\Rightarrow -(2a + 2b - ab) = 2\sqrt{a^2 + b^2}$$

('–' sign is used as (2, 2) lies below the line $bx + ay - ab = 0$)



If $P(h, k)$ is the midpoint of AB, then $a = 2h$, $b = 2k$

Thus, $4kh - 4h - 4k = 2\sqrt{4h^2 + 4k^2}$

Hence, locus of 'P' is,

$$x + y - xy + \sqrt{x^2 + y^2} = 0$$

$\therefore \lambda = 1$

25. (a) : Let $P \equiv (h, k)$

Tangent in terms of m is $y = mx + \frac{a}{m}$

$\Rightarrow m^2x - my + a = 0$

If it passes through (h, k), then

$m^2h - mk + a = 0$.

Since, angle between these tangents is $\pi/4$.

$$\therefore \tan\frac{\pi}{4} = \frac{|m_1 - m_2|}{|1 + m_1m_2|}$$

$\Rightarrow (m_1 - m_2)^2 = (1 + m_1m_2)^2$

$\Rightarrow (m_1 + m_2)^2 - 4m_1m_2 = (1 + m_1m_2)^2$

$$\Rightarrow \frac{k^2}{h^2} - \frac{4a}{h} = \left(1 + \frac{a}{h}\right)^2$$

Thus, required locus is, $y^2 = 4ax + (a + x)^2$

26. (c) : Tangent in terms of 'm' for $y^2 = 4x$ is $y = mx + \frac{1}{m}$

and for $\frac{x^2}{8} + \frac{y^2}{2} = 1$ tangent is, $y = mx \pm \sqrt{8m^2 + 2}$

For common tangents, we must have $\frac{1}{m} = \pm\sqrt{8m^2 + 2}$

$$\Rightarrow m = \pm\frac{1}{2}$$

Thus, common tangents are $y = \frac{x}{2} + 2$ and $y = -\frac{x}{2} - 2$

i.e. $2y - x - 4 = 0$ and $2y + x + 4 = 0$

27. (c) : Length of latus rectum $= \frac{2b^2}{a}$

For $xy = c^2$, $a = b$ and $\frac{a^2}{2} = c^2$

$\therefore$ Required length $= 2 \cdot \frac{a^2}{a} = 2a = 2\sqrt{2}c$

28. (d) : Given line is $2x = 3y = 4z$

i.e. $\frac{x}{6} = \frac{y}{4} = \frac{z}{3}$

Let the plane be $ax + by + cz = 1$

$\therefore 6a + 4b + 3c = 0$, $2a - b = 1$ and $3a - 4b + 5c = 1$

$$\Rightarrow a = \frac{29}{85}, b = -\frac{27}{85}, c = -\frac{22}{85}$$

Thus, equation of plane is $29x - 27y - 22z = 85$.

29. (c) : Given line passes through $P(1, 2, 4)$ and this point also lies on the given plane.

Thus, required line will be in the form of

$$\frac{x - 1}{l} = \frac{y - 2}{m} = \frac{z - 4}{n}$$

Any point on the given line is

$(-r_1 + 1, 3r_1 + 2, r_1 + 4)$

If $r_1 = 1$, this point becomes $P \equiv (0, 5, 5)$.

Let $Q(a, b, c)$ be the reflection of 'P' in the given plane, then

$$\frac{a}{2} + \frac{b + 5}{2} + \frac{5 + c}{2} = 7$$

i.e., $a + b + c = 4$ and $\frac{a}{1} = \frac{b - 5}{1} = \frac{c - 5}{1} = \lambda$ (say)

MAGAZINE KING

$\Rightarrow$ $a = \lambda$, $b = 5 + \lambda$, $c = 5 + \lambda$
$\Rightarrow$ $10 + 3\lambda = 4 \Rightarrow \lambda = -2$.
Thus, $Q \equiv (-2, 3, 3)$
Hence, direction ratios of reflected line are $-3, 1, -1$.
Thus, its equation is $\frac{x-1}{-3} = \frac{y-2}{1} = \frac{z-4}{-1}$.

30. (b) : $\frac{\sin x - \sin z}{\cos z - \cos x} = \frac{2\cos\left(\frac{x+z}{2}\right).\sin\left(\frac{x-z}{2}\right)}{2\sin\left(\frac{x+z}{2}\right).\sin\left(\frac{x-z}{2}\right)}$

$= \cot\left(\frac{x+z}{2}\right) = \cot(y)$

31. (c) : We have, $27^{\cos x} + 81^{\sin x}$

$= 3^{3\cos x} + 3^{4\sin x} \geq 2 \cdot \sqrt{3^{3\cos x} \cdot 3^{4\sin x}}$

$= 2 \cdot 3^{\frac{1}{2}(3\cos x + 4\sin x)} \geq 2 \cdot 3^{-\frac{5}{2}} = \frac{2}{9\sqrt{3}}$

32. (c) : $\sin^{-1}\left(\frac{\sqrt{r} - \sqrt{r-1}}{\sqrt{r(r+1)}}\right) = \tan^{-1}\left(\frac{\sqrt{r} - \sqrt{r-1}}{1 + \sqrt{r(r-1)}}\right)$

$= \tan^{-1}\sqrt{r} - \tan^{-1}(\sqrt{r-1})$

$\Rightarrow \sum_{r=1}^{n} \sin^{-1}\left(\frac{\sqrt{r} - \sqrt{r-1}}{\sqrt{r(r+1)}}\right) = \sum_{r=1}^{n}\left(\tan^{-1}\sqrt{r} - \tan^{-1}\sqrt{r-1}\right)$

$= \tan^{-1}\sqrt{n}$

33. (b) : Clearly, the direction ratios of normal to the plane are $4, -3, -2$.
Thus equation of plane will be $4x - 3y - 2z = d$
It will necessarily pass through $(3, -2, -1)$
i.e., $d = 12 + 6 + 2 = 20$
Thus the equation of plane is $4x - 3y - 2z = 20$.

34. (d) : $\angle A = \pi/2 \Rightarrow 2R = a$

Also, $r = (s - a)\tan\frac{A}{2} = (s-a) = \frac{1}{2}(b + c - a)$

$\therefore \quad r + 2R = \frac{1}{2}(a + b + c)$

35. (b) : Clearly, $a + b > c$

$\Rightarrow (\sqrt{a} + \sqrt{b})^2 - 2\sqrt{ab} > (\sqrt{c})^2$

$\Rightarrow (\sqrt{a} + \sqrt{b})^2 > (\sqrt{c})^2 \Rightarrow \sqrt{a} + \sqrt{b} > \sqrt{c}$

$\Rightarrow \sqrt{a} + \sqrt{b} - \sqrt{c} > 0$

36. (a) : Let $I = \int \tan^{-1}\sqrt{x}\, dx$

$= x \cdot \tan^{-1}\sqrt{x} - \int \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} \cdot x\, dx$

$= x \cdot \tan^{-1}\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}}{1+x} dx$

Now, putting $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt$
$\Rightarrow dx = 2t\, dt$, we get

$I = x \cdot \tan^{-1}\sqrt{x} - \frac{1}{2}\int \frac{t}{1+t^2} 2t\, dt$

$= x\tan^{-1}\sqrt{x} - \int \frac{t^2 + 1 - 1}{1+t^2} dt$

$= x\tan^{-1}\sqrt{x} - t + \tan^{-1} t + C$

$= x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}\sqrt{x} + C$

37. (d) : Clearly, the given curves intersect for $a \in R - \{0\}$.

For the parabola, $\frac{dy}{dx} = \frac{8}{y}$. For the ellipse, $\frac{dy}{dx} = \frac{-4x}{a^2 y}$

For the curves to intersect orthogonally,

$\left(\frac{8}{y}\right)\left(-\frac{4x}{a^2 y}\right) = -1$

$\Rightarrow a^2y^2 = 32x \Rightarrow 16a^2x = 32x \Rightarrow a^2 = 2 \Rightarrow a = \pm\sqrt{2}$

38. (c) : We have, $f'(x) = 2e^x + ae^{-x} + 2a + 1$

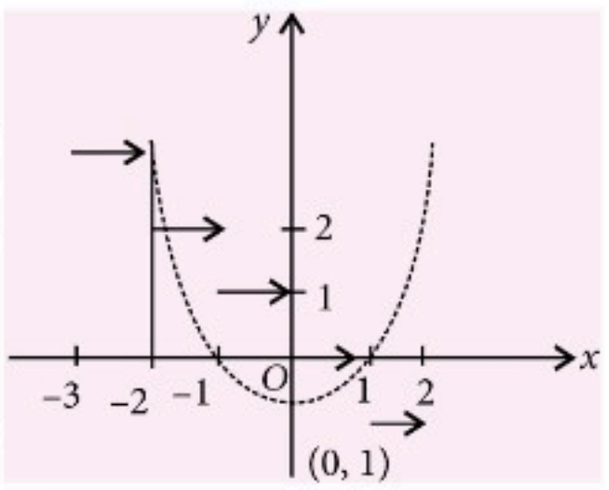
$= e^{-x}(2e^{2x} + (2a + 1)e^x + a)$

$= 2e^{-x}\left(e^{2x} + \left(a + \frac{1}{2}\right)e^x + \frac{a}{2}\right)$

$= 2e^{-x}(e^x + a)\left(e^x + \frac{1}{2}\right)$

For $f(x)$ to be increasing, $f'(x) \geq 0\ \forall\ x \in R$
$\Rightarrow e^x + a \geq 0\ \forall\ x \in R \Rightarrow a \geq 0$

39. (d) : We must have $[x] - 1 + x^2 \geq 0$
$\Rightarrow x^2 - 1 \geq -[x]$
The graphs of $y = x^2 - 1$ and $y = -[x]$ intersect somewhere in $(-2, -1)$. Thus at the point of intersection, $x^2 - 1 = 2$
$\Rightarrow x = -\sqrt{3}$
From the adjacent graph, we get domain as $(-\infty, -\sqrt{3}] \cup [1, \infty)$.



40. (a) : $\frac{r}{(r+1)!} = \frac{r+1-1}{(r+1)!} = \frac{1}{r!} - \frac{1}{(r+1)!}$

$= -\left(\frac{1}{(r+1)!} - \frac{1}{r!}\right) = -(V(r) - V(r-1))$

$\sum_{r=1}^{n} \frac{r}{(r+1)!} = -(V(n) - V(0)) = 1 - \frac{1}{(n+1)!}$

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JEE WORK OUTS

PAPER-1

SECTION-I

Integer Answer Type

This section contains eight questions. The answer to each question is a single digit integer ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme : +4 If the bubble corresponding to the answer is darkened, 0 In all other cases.

1. Let $f: R \to R$ be a differentiable function satisfying $f(x) = f(y)\, f(x - y)\ \forall\ x, y \in R$ and $f'(0) = \int_0^4 \{2x\}\, dx$, where $\{.\}$ denotes the fractional part function and $f'(-3) = \alpha e^{\beta}$. Then, $|\alpha + \beta|$ is equal to _____.

2. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors, then find the maximum value of $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2$

3. If $\int \frac{\ln x}{x^3}\, dx = -\frac{\ln x}{ax^2} - \frac{1}{bx^2} + c$, then the value of $a + b$ must be _____.

4. TP and TQ are any two tangents to a parabola and the tangent at a third point R cuts them in P' and Q' respectively, then the value of $\frac{TP'}{TP} + \frac{TQ'}{TQ}$ must be ____.

5. If $\alpha, \beta, \gamma, \delta$ are the solutions of the equation $\tan\left(\theta + \frac{\pi}{4}\right) = 3\tan 3\theta$, no two of which have equal tangents, then the value of $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$ is equal to _____.

6. The number of solutions for the equation $\log_4 (2x^2 + x + 1) - \log_2 (2x - 1) = 1$, is _____.

7. Find the value of

$$4\left(\cos\frac{2\pi}{15} + \cos\frac{4\pi}{15} - \cos\frac{7\pi}{15} - \cos\frac{\pi}{15}\right)$$

8. If $(y^2 - 5y + 3)(x^2 + x + 1) < 2x$ for all $x \in R$, then number of integral values of y will be

SECTION-II

One or More Than One Option Correct Type

This section contains ten questions. Each question has four options (a), (b), (c) and (d). One or More than one of these four option (s) is (are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened, 0 If none of the bubbles is darkened, –2 In all other cases.

9. If a, b, c are real numbers and z is a complex number such that $a^2 + b^2 + c^2 = 1$ and $b + ic = (1 + a)z$, then $\frac{1+iz}{1-iz}$ equals

(a) $\frac{b-ic}{1-ia}$ (b) $\frac{a+ib}{1+c}$ (c) $\frac{1-c}{a-ib}$ (d) $\frac{1+a}{b+ic}$

10. If the quadratic polynomial $P(x) = (p - 3)x^2 - 2px + 3p - 6$ ranges from $[0, \infty)$ for every $x \in R$, then the value of p can be

(a) $\frac{3}{2}$ (b) 4 (c) 6 (d) 7

11. All non-zero complex numbers on the complex plane satisfying $z + \frac{1}{z} \in R$ can lie on

(a) $y = 0$ (b) $x = 0$
(c) $y = x$
(d) a unit circle with centre at the origin

12. Find the domain of $f(x) = \sin^{-1}[2 - 4x^2]$ ([.] denotes the greatest integer function)

(a) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right] - \{0\}$ (b) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$

(c) $\left[-\frac{\sqrt{3}}{2}, 0\right)$ (d) $\left[-\frac{\sqrt{3}}{2}, 8\right]$

13. $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$ is equal to

(a) $\frac{n(n+1)(n+2)}{6}$ (b) $\sum_{n=1}^{n} n^3$

(c) $\frac{n(n+1)(2n+1)}{6}$ (d) ${}^{n+2}C_3$

14. The line $3x + 6y = k$ intersect the curve $2x^2 + 2xy + 3y^2 = 1$ at points A and B. The circle on AB as diameter passes through the origin. The possible value(s) of k is (are)

(a) 3 (b) 4 (c) -4 (d) -3

15. Let $P(x) = x^2 + bx + c$, where b and c are integers. If $P(x)$ is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, then

(a) $P(x) = 0$ has imaginary roots

(b) $P(x) = 0$ has roots of opposite sign

(c) $P(1) = 4$ (d) $P(1) = 6$

16.

$$\left[({}^nC_0 + {}^nC_3 + ...) - \frac{1}{2}({}^nC_1 + {}^nC_2 + {}^nC_4 + {}^nC_5 + ...)\right]^2 + \frac{3}{4}({}^nC_1 - {}^nC_2 + {}^nC_4 - {}^nC_5 + ...)^2 =$$

(a) 3 (b) 4 (c) 2 (d) 1

17. For every pair of continuous functions $f, g : [0, 1] \to R$, such that $\max\{f(x) : x \in [0, 1]\} = \max\{g(x) : x \in [0, 1]\}$, the correct statement(s) is (are):

(a) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$

(b) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$

(c) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0, 1]$

(d) $(f(c))^2 = (g(c))^2$ for some $c \in [0, 1]$

18. Let $\overrightarrow{PR} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\overrightarrow{SQ} = \hat{i} - 3\hat{j} - 4\hat{k}$ determine diagonals of a parallelogram $PQRS$ and $\overrightarrow{PT} = \hat{i} + 2\hat{j} + 3\hat{k}$ be another vector. Then the volume of the parallelepiped determined by the vectors $\overrightarrow{PT}$, $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ is

(a) 5 (b) 20 (c) 10 (d) 30

SECTION-III

Matrix Match Type

This section contains 2 questions. Each question contains two columns, Column I and Column II. Column I has four entries (A), (B), (C) and (D). Column II has four entries (p), (q), (r) and (s). Match the entries in Column I with the entries in Column II. One or more entries in Column I may match with one or more entries in Column II. The ORS contains a 4 × 5 matrix whose layout will be similar to the one shown. For each entry in Column I, darken the bubbles of all the matching entries. For example, if entry (A) in Column I matches with entries (q), (r) and (s), then darken these three bubbles in the ORS. Similarly, for entries (B), (C) and (D). Marking scheme : For each entry in Column I. +2 If only the bubble(s) corresponding to all the correct match(es) is (are) darkened, 0 If none of the bubbles is darkened, –1 In all other cases.

	p	q	r	s
A	(p)	(q)	(r)	(s)
B	(p)	(q)	(r)	(s)
C	(p)	(q)	(r)	(s)
D	(p)	(q)	(r)	(s)

19. Match the following: -

Column-I		Column-II	
(A)	Consider the differential equation, $y = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ It's degree is	(p)	1
(B)	The differential equation representing the family of curve $y^2 = 2c(x + \sqrt{c})$, where c is a parameter, is of degree	(q)	4
(C)	Consider differential equation, $1 + \left(\frac{dy}{dx}\right) + \frac{1}{2!}\left(\frac{dy}{dx}\right)^2 + \frac{1}{3!}\left(\frac{dy}{dx}\right)^3 + ...\infty = y$ then its degree is	(r)	2
(D)	Consider the differential equation, $\frac{dy}{dx} = \left(1 + \left(\frac{d^2y}{dx^2}\right)^2\right)^{2/3}$ Its degree is	(s)	3

20. Consider the system of equations

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

MAGAZINE KING

Now, let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

According to Crammer's rule,

$$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$$

Match the following: -

Column-I		Column-II	
(A)	If $\Delta \neq 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$, then the solutions are	(p)	Unique
(B)	If $\Delta_1 = \Delta_2 = \Delta_3 = \Delta = 0$, then the solutions are	(q)	Inconsistent
(C)	If $\Delta \neq 0$ and atleast one of Δ_1, Δ_2, Δ_3 is non-zero, then the solutions are	(r)	Non-Trivial
(D)	If atleast one of Δ_1, Δ_2, Δ_3 is non-zero, and $\Delta = 0$, then the solutions are	(s)	Infinitely many

PAPER-2

SECTION-I

Integer Answer Type

This section contains eight questions. The answer to each question is a single digit integer ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme : +4 If the bubble corresponding to the answer is darkened, 0 In all other cases.

1. Let $f(x, y)$ be a periodic function satisfying the condition $f(x, y) = f((2x + 2y), (2y - 2x))\ \forall\ x, y \in R$. Now, define a function g by $g(x) = f(2^x, 0)$. If period of $g(x)$ is α, then value of $|\alpha - 7|$ is

2. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \leq d_1(P) + d_2(P) \leq 4$, is

3. The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a triangle which has two of its sides along the coordinate axes. The locus of the circumcentre of the triangle is $x + y - xy + k\sqrt{x^2 + y^2} = 0$, find k.

4. If $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ for all $x \in R$, then find the value of $f(1)$.

5. Find $\lim\limits_{x \to \infty} \left| \dfrac{(3x^4 + 2x^2)\sin\frac{1}{x} + |x|^2 + 5}{|x|^3 + |x|^2 + |x| + 1} \right|$

6. If $y(t)$ is a solution of $(1+t)\dfrac{dy}{dt} - ty = 1$ and $y(0) = -1$, then $y(1) + \dfrac{3}{2}$ is equal to

7. Suppose that f is differentiable for all x and that $f'(x) \leq 2$ for all x. If $f(1) = 2$ and $f(4) = 8$, then $f(2)$ has the value equal to_____.

8. Two whole numbers are randomly selected and multiplied. If the probability that the units place in their product is "Even" is p and the probability that the units place in their product is "odd" is q, then $\left(\dfrac{p}{q}\right) + 1$ is equal to

SECTION-II

One or More Than One Option Correct Type

This section contains eight questions. Each question has four options (a), (b), (c) and (d). One or More than one of these four option (s) is (are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened, 0 If none of the bubbles is darkened, –2 In all other cases.

9. The point of intersection of the plane $\vec{r}.(3\hat{i} - 5\hat{j} + 2k) = 6$ with the straight line passing through the origin and perpendicular to the plane $2x - y - z = 4$ is

(a) $(1, -1, -1)$ (b) $(-1, -1, 2)$

(c) $(4, 2, 2)$ (d) $\left(\dfrac{4}{3}, \dfrac{-2}{3}, \dfrac{-2}{3}\right)$

10. If the median through A of a ΔABC having vertices $A \equiv (2, 3, 5)$, $B \equiv (-1, 3, 2)$ and $C \equiv (\lambda, 5, \mu)$ is equally inclined to the axes, then

(a) $\lambda = 7$ (b) $\mu = 10$

(c) $\lambda = 10$ (d) $\mu = 7$

11. The integral of $\dfrac{1}{\sin^2 x + \tan^2 x}$ must be

(a) $-\dfrac{1}{2}\left[\tan x + \dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{1}{\sqrt{2}}\tan x\right)\right] + c$

(b) $-\dfrac{1}{2}\left[\cot x + \dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{1}{\sqrt{2}}\tan x\right)\right] + c$

(c) $-\frac{1}{2}\left[\cot x-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan x\right)\right]+c$

(d) $-\frac{1}{2}\left[\cot x-\frac{1}{\sqrt{2}}\cot^{-1}\left(\frac{1}{\sqrt{2}}\tan x\right)\right]+c$

12. $I=\int\sqrt{\frac{\cos^3 x}{\sin^{11} x}}\,dx$ is equal to

(a) $-\frac{2}{5}\cot^{3/2} x-\frac{2}{9}\cot^{9/2} x+c$

(b) $-\frac{2}{5}\cot^{5/2} x-\frac{2}{9}\cot^{9/2} x+c$

(c) $-\frac{2}{5}\cot^{5/2} x-\frac{2}{9}\cot^{3/2} x+c$

(d) None of these

13. The line $y = x + 5$ touches

(a) the parabola $y^2 = 20x$

(b) the ellipse $9x^2 + 16y^2 = 144$

(c) the hyperbola $\frac{x^2}{29}-\frac{y^2}{4}=1$

(d) the circle $x^2 + y^2 = 25$

14. How many numbers between 5,000 and 10,000 can be formed using the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 each digit appearing not more than once in each number ?

(a) 1680 (b) $5 \times {}^8P_8$

(c) $30 \times {}^8C_3$ (d) $5! \times {}^8C_3$

15. Triplets (p, q, r) is chosen from a set {1, 2, 3, 4, 5, 6, 7, 8, 9} such that $x < y \le z$. There could be …… such triplets.

(a) 143 (b) 200 (c) 180 (d) 120

16. If in triangle ABC, CD is the angle bisector of the angle ACB, then CD is equal to

(a) $\frac{a+b}{2ab}\cos\frac{C}{2}$ (b) $\frac{a+b}{ab}\cos\frac{C}{2}$

(c) $\frac{2ab}{a+b}\cos\frac{C}{2}$ (d) $\frac{b\sin A}{\sin\left(B+\frac{C}{2}\right)}$

SECTION-III

Comprehension Type

This section contains 2 paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each of these questions has 4 choices (a), (b), (c) and (d), out of which only one is correct. Correct answer 3 marks and wrong answer –1.

Paragraph for Question. No. 17 and 18

Consider a hyperbola whose centre is at origin. A line $x + y = 2$ touches this hyperbola at $P(1, 1)$ and intersect the asymptotes at A and B such that $AB = 6\sqrt{2}$ units.

17. Equation of asymptotes are

(a) $2x^2 + 5xy + 2y^2 = 0$ (b) $2x^2 - 5xy - 2y^2 = 0$

(c) $2x^2 - 5xy + 2y^2 = 0$ (d) $2x^2 + 5xy - 2y^2 = 0$

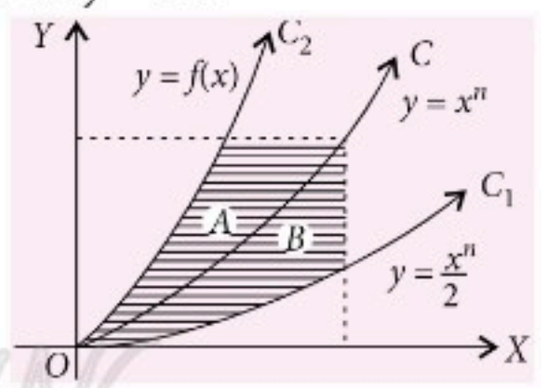
18. Equation of tangent to the hyperbola at $\left(-1,\frac{7}{2}\right)$ is

(a) $3x + 2y = 2$ (b) $3x + 4y = 11$

(c) $3x + 2y = 4$ (d) $3x + 2y = 7$

Paragraph for Question. No. 19 and 20

Let C_1 and C_2 be two curves passing through the origin as shown in the figure. A curve C is said to "bisect the area" between C_1 and C_2, if for each point P of C, the two shaded regions A and B shown in the figure have equal areas. Let the equation of the curve C_1 be $y=\frac{x^n}{2}$ and that of C be $y = x^n$.



19. Area of the total shaded region $(A + B)$, if x– coordinate of P is x and $n = 2$ is

(a) $\frac{x^2}{2}$ (b) $\frac{x^3}{2}$ (c) $\frac{x^3}{3}$ (d) $\frac{x^3}{6}$

20. Equation of the curve C_2 for $n = 2$ is $y =$

(a) $\frac{3x^3}{4}$ (b) $\frac{4x^2}{3}$ (c) $\frac{9x^2}{16}$ (d) $\frac{16x^2}{9}$

ANSWER KEY

Paper-1

1.	(4)	**2.**	(9)	**3.**	(6)	**4.**	(1)
5.	(0)	**6.**	(1)	**7.**	(2)	**8.**	(2)
9.	(b, c)	**10.**	(c)	**11.**	(a, d)	**12.**	(a)
13.	(a, d)	**14.**	(a, d)	**15.**	(a, c)	**16.**	(d)
17.	(a, d)	**18.**	(c)				

19. (A) → (r), (B) → (s), (C) → (p), (D) → (q)

20. (A) → (p), (B) → (r, s), (C) → (p, r), (D) → (q)

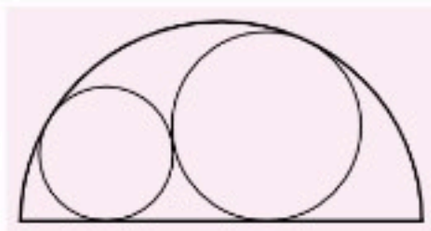
Paper-2

1.	(5)	**2.**	(6)	**3.**	(1)	**4.**	(4)
5.	(2)	**6.**	(1)	**7.**	(4)	**8.**	(4)
9.	(d)	**10.**	(a, b)	**11.**	(b, d)	**12.**	(b)
13.	(a, b, c)	**14.**	(a, c)	**15.**	(d)	**16.**	(c, d)
17.	(a)	**18.**	(c)	**19.**	(c)	**20.**	(d)

OLYMPIAD CORNER

1. Two externally tangent circles of radii R_1 and R_2 are internally tangent to a semicircle of radius 1, as in the figure. Prove that

$R_1 + R_2 \le 2(\sqrt{2}-1)$.



2. Ten players participate in a ping-pong tournament, in which every two players play against each other exactly once. If player i beats player j, player j beats player k, and player k beats player i, then the set $\{i, j, k\}$ is called a triangle. Let w_i and l_i denote the number of games won and lost, respectively, by the i^{th} player. Suppose that, whenever i beats j, then $l_i + w_j \ge 8$. Prove that there are exactly 40 triangles in the tournament.

3. In an equilateral triangle ABC (of side length 2) consider the incircle T.

(a) Show that for all points P of T,

$\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 = 5$.

(b) Show that for all points P of T, it is possible to construct a triangle of sides PA, PB, PC, with area $\sqrt{3}/4$.

4. Are there any nonconstant differentiable functions $f: R \to R$ such that $f(f(f(x))) = f(x) \ge 0$, for all $x \in R$?

5. **(a)** Find all positive integers $p \le q \le r$ satisfying the equation $p + q + r + pq + qr + rp = pqr + 1$.

(b) For each such solution (p, q, r), evaluate $\tan^{-1}(1/p) + \tan^{-1}(1/q) + \tan^{-1}(1/r)$.

6. Let n be a fixed positive integer. Show that for any nonnegative integer k, the equation

$x_1^3 + x_2^3 + \ldots + x_n^3 = y^{3k+2}$

has infinitely many solutions in positive integers x_i and y.

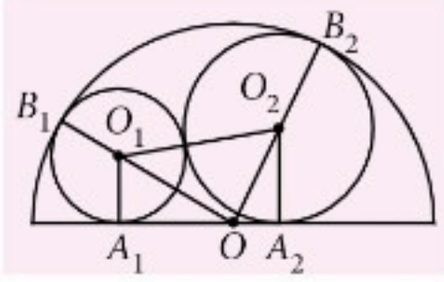
7. Suppose that u is a real parameter with $0 < u < 1$. Define

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le u \\ 1-(\sqrt{ux} + \sqrt{(1-u)(1-x)})^2 & \text{if } 0 \le x \le 1 \end{cases}$$

and define the sequence $\{u_n\}$ recursively as follows: $u_1 = f(1)$, and $u_n = f(u_{n-1})$ for all $n > 1$.
Show that there exists a positive integer k for which $u_k = 0$.

SOLUTIONS

1. Let O_1, O_2, and O denote the centres of the circles, and let A_1, A_2, B_1 and B_2 denote the points of tangency of these circles with the semicircle, as shown in the diagram. Then $O_1O_2 = R_1 + R_2$, $O_1A_1 = R_1$, and $O_2A_2 = R_2$. So,



$$A_1A_2 = \sqrt{(O_1O_2)^2 - (O_1A_1 - O_2A_2)^2}$$

$$= \sqrt{(R_1+R_2)^2 - (R_1-R_2)^2} = 2\sqrt{R_1R_2}.$$

Also, $OB_1 = 1$, $OB_2 = 1$, $O_1B_1 = R_1$ and $O_2B_2 = R_2$, so that $OO_1 = 1 - R_1$ and $OO_2 = 1 - R_2$.
Therefore, $A_1A_2 = OA_1 + OA_2$

$$= \sqrt{(OO_1)^2 - (O_1A_1)^2} + \sqrt{(OO_2)^2 - (O_2A_2)^2}$$

$$= \sqrt{(1-R_1)^2 - R_1^2} + \sqrt{(1-R_2)^2 - R_2^2} = \sqrt{1-2R_1} + \sqrt{1-2R_2}.$$

Thus, $\sqrt{1-2R_1} + \sqrt{1-2R_2} = 2\sqrt{R_1R_2}$.
Squaring, then dividing each term by 2 and rearranging the terms, we get

$$\sqrt{(1-2R_1)(1-2R_2)} = 2R_1R_2 + R_1 + R_2 - 1.$$

Square both sides and simplify:

$8R_1R_2 = (2R_1R_2 + R_1 + R_2)^2$, ...(i)

So, $2\sqrt{2R_1R_2} = 2R_1R_2 + R_1 + R_2$ ...(ii)

Thus, $R_1 + R_2 = 2\sqrt{R_1R_2}(\sqrt{2} - \sqrt{R_1R_2})$

$\leq (R_1 + R_2)(\sqrt{2} - \sqrt{R_1R_2})$, and therefore $\sqrt{R_1R_2} \leq \sqrt{2} - 1$.

Now, consider the function $f(x) = 2x\ (\sqrt{2} - x)$. $f(x)$ is increasing on the interval $(0, 1/\sqrt{2})$ since $f'(x) = 2\sqrt{2} - 4x > 0$ for x in the interval. Since $0 < \sqrt{R_1R_2} \leq \sqrt{2} - 1 < 1/\sqrt{2}$ and $R_1 + R_2 = f(\sqrt{R_1R_2})$ from (ii), $R_1 + R_2$ attains its maximum when $\sqrt{R_1R_2} = \sqrt{2} - 1$.

Hence, $R_1 + R_2 \leq 2(\sqrt{2} - 1)[\sqrt{2} - (\sqrt{2} - 1)] = 2(\sqrt{2} - 1)$.

2. Let P_i denote the i^{th} player $i = 1, 2, \ldots, 10$. Without loss of generality, assume that $w_1 \leq w_2 \leq \ldots \leq w_{10}$. Let $M = \max_{1 \leq i \leq 10} w_i = w_{10}$ and $m = \min_{1 \leq i \leq 10} w_i = w_1$. Since $w_i + l_i = 9$ for all i, the condition $l_i + w_j \geq 8$, whenever P_i beats P_j, is equivalent to

$w_i \leq w_j + 1$. ...(i)

Clearly $\sum_{i=1}^{10} w_i = \binom{10}{2} = 45$. ...(ii)

We first show that

$(w_1, w_2, \ldots, w_{10}) = (4, 4, 4, 4, 4, 5, 5, 5, 5, 5)$

Since P_1 beats m players and loses to $9 - m$ others, we get from (i) and (ii) that $45 \leq m + Mm + (9 - m)(m + 1)$ or $36 \leq (M - m + 9)m$. ...(iii)

Similarly, since P_{10} beats M players and loses to $9 - M$ others, we have $45 \geq M + M(M - 1) + (9 - M)m = M^2 + (9 - M)m$. ...(iv)

Since $M \leq 9$, from (iii) we obtain $36 \leq (18 - m)m$, which implies $m \geq 3$. From (iii) we get $M^2 \leq 45$, *i.e.*, $M \leq 6$. If $M = 6$, then from (iv), $3m \leq 9$ or $m \leq 3$ and so $m = 3$. Similarly, if $m = 3$ then from (iii) $M \geq 6$ and so $M = 6$. Since 10×45, $M \neq m$. Therefore, the only possibilities are $(M, m) = (6, 3)$ and $(5, 4)$.

Note : At this point, Lau listed all 10-tuples $(w_1, w_2, \ldots, w_{10})$ satisfying $(M, m) = (6, 3)$ and (2), and then showed that none were possible. But the case $(M, m) = (6, 3)$ can be eliminated as follows. If $m = w_1 = 3$, then six players beat P_1, so by (i) at least seven of the w_i's (including w_1) are ≤ 4. If $M = w_{10} = 6$, then P_{10} beats six players, so by (i) atleast seven of the w_i's (including w_{10}) are ≥ 5. But since there are only ten w_i's this is impossible.

Thus $(M, m) = (5, 4)$, and $(4, 4, 4, 4, 4, 5, 5, 5, 5, 5)$ is the only possible solution. By exhibiting the tournament graphically, one can see that this is indeed a solution. Now, there is a theorem in graph theory which states that the number of transitive triples [3 players P_i, P_j and P_k such that P_i beats P_j, P_j beats P_k, and P_i beats P_k – *Ed.*] in a tournament with n vertices and score sequence $(w_1, w_2, \ldots, w_n)$ is $\sum_{i=1}^{n} w_i (w_i - 1)/2$.

Note : This result is true because the number of such triples is just the number of unordered pairs $\{P_j, P_k\}$ of (distinct) players beaten by P_i, which is $\binom{w_i}{2}$, added up over all P_i.] It follows that, with $n = 10$, the number of triangles (cyclic triples) is

$$\binom{10}{3} - \frac{1}{2}\sum_{i=1}^{n} w_i(w_i - 1) = 120 - \frac{1}{2}(5)(5)(5-1) - \frac{1}{2}(5)(4)(4-1)$$

$= 40$.

This completes the proof.

SAMURAI SUDOKU

Samurai Sudoku puzzle consists of five overlapping sudoku grids. The standard sudoku rules apply to each 9 × 9 grid. Place digits from 1 to 9 in each empty cell. Every row, every column and every 3 × 3 box should contain one of each digit.

The puzzle has a unique answer.

		5			8			
	3			9			5	
4		6					1	7
5			3	6			2	
			8		2	5		1
	8			5	7	3		
		7	4					
			7	3				
	1	9			6			

				5			4	
		3			4			9
	4	8				3	6	
		1	4		8	6		
6	3		2	1				
4				3	6			1
			9				3	
			1					
				7			1	4

				8				
				9	5			
				3				
			4		9			
			2		6			
			1		5			
					3			
					1			
					5			

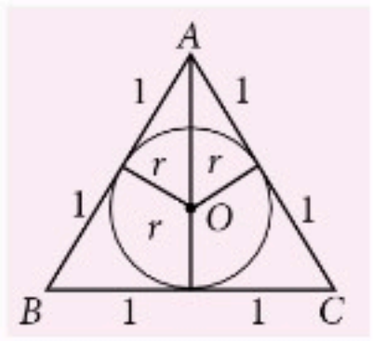
		2	1					
	7	1			9			
			2	6				
8			4	1			7	
	4			8	3	1		
			5		2	8		6
	8			3			6	
		7			1			
3		9					8	5

				6				3
					8			2
					4			
			5	9		6	1	
				8		4	1	
				4	1			
			4			9		
					6			5
				3	9			

				6				3
					8			2
					4			
			5	9		6	1	
				8		4	1	
				4	1			
			4			9		
					6			5
				3	9			

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

3. Take O as origin, with the x-axis parallel to BC, and y-axis along OA.



Then the incircle has radius $r = 1/\sqrt{3}$ and A, B, C have coordinates $A\left(0, \frac{2}{\sqrt{3}}\right), B\left(-1, \frac{-1}{\sqrt{3}}\right), C\left(1, \frac{-1}{\sqrt{3}}\right)$.

A point on the incircle has coordinates parameterized by θ, $0 \le \theta < 2\pi$ given by $P\left(\frac{1}{\sqrt{3}}\cos\theta, \frac{1}{\sqrt{3}}\sin\theta\right)$.

(a) Let $c = \cos\theta$, $s = \sin\theta$. Then

$AP^2 + BP^2 + CP^2$

$$= \left[\frac{1}{3}c^2 + \frac{1}{3}(2-s)^2\right] + \left[\frac{1}{3}(c+\sqrt{3})^2 + \frac{1}{3}(s+1)^2\right] \left[\frac{1}{3}(c-\sqrt{3})^2 + \frac{1}{3}(s+1)^2\right]$$

$$= \frac{1}{3}[c^2 + 4 - 4s^2 + s^2 + c^2 + 2\sqrt{3}c + 3 + s^2 + 2s + 1 + c^2 - 2\sqrt{3}c + 3 + s^2 + 2s + 1]$$

$$= \frac{1}{3}[3(c^2 + s^2) + 12] = 5$$

since $c^2 + s^2 = 1$.

(b) Now, set $x = AP = \frac{1}{\sqrt{3}}\sqrt{5-4s}$,

$y = BP = \frac{1}{\sqrt{3}}\sqrt{5+2s+2\sqrt{3}c}$, and

$z = CP = \frac{1}{\sqrt{3}}\sqrt{5+2s-2\sqrt{3}c}$.

By reflection and rotational geometry, the distances AP, BP, CP will be a permutation of those obtained when $\pi/6 \le \theta \le \pi/2$, so that $1/\sqrt{3} \le x \le 1$.

Also,

$$y = \frac{1}{\sqrt{3}}\sqrt{5+4\left(\frac{1}{2}s + \frac{\sqrt{3}}{2}c\right)} = \frac{1}{\sqrt{3}}\sqrt{5+4\cos\left(\theta - \frac{\pi}{6}\right)}$$

so that $\sqrt{7}/\sqrt{3} \le y \le \sqrt{3}$ and

$$z = \frac{1}{\sqrt{3}}\sqrt{5+4\left(\frac{1}{2}s - \frac{\sqrt{3}}{2}c\right)} = \frac{1}{\sqrt{3}}\sqrt{5+4\sin\left(\theta - \frac{\pi}{3}\right)}$$

so that $\sqrt{5}/\sqrt{3} \le z \le \sqrt{7}/\sqrt{3}$. Thus,

$$(x+y)_{\min} = \frac{1+\sqrt{7}}{\sqrt{3}} > z_{\max} = \frac{\sqrt{7}}{\sqrt{3}}$$

and $(x+z)_{\min} = \frac{\sqrt{7}+\sqrt{5}}{\sqrt{3}} > x_{\max} = 1$

and $(z+x)_{\min} = \frac{1+\sqrt{5}}{\sqrt{3}} > y_{\max} = \sqrt{3}$,

and x, y, z (*i.e.* AP, BP, CP) can form the sides of a triangle.

From Heron's formula the area, F, of this triangle is given by

$$F^2 = \frac{1}{2}(x+y+z)\frac{1}{2}(-x+y+z)\cdot\frac{1}{2}(x-y+z)\cdot\frac{1}{2}(x+y-z)$$

$$= \frac{1}{16}[(y+z)^2 - x^2][x^2 - (y-z)^2]$$

$$= \frac{1}{16}[\frac{1}{3}(5+2s+2\sqrt{3}c+5+2s-2\sqrt{3}c-5+4s)$$

$$+2\cdot\frac{1}{3}(5+2s+2\sqrt{3}c)^{1/2}(c+2s-2\sqrt{3}c)^{1/2}]$$

$$\cdot[\frac{1}{3}(5-4s-5-2s-2\sqrt{3}c-5-2s+2\sqrt{3}c)$$

$$+2\cdot\frac{1}{3}(5+2s+2\sqrt{3}c)^{1/2}(5+2s-2\sqrt{3}c)^{1/2}]$$

$$= \frac{1}{48}\left[(5+8s) + 2\sqrt{(5+2s)^2 - 12c^2}\right]$$

$$\left[-(5+8s) + 2\sqrt{(5+2s)^2 - 12c^2}\right]$$

$$= \frac{1}{48}[100 + 80s + 16s^2 - 48c^2 - 25 - 80s - 64s^2]$$

$= \frac{1}{48}[75-48] = \frac{9}{16}$, since $c^2 + s^2 = 1$. Thus $F = 3/4$.

4. Applying f to both sides of the functional equation

$$f(f(f(x))) = f(x) \ge 0 \quad \ldots(i)$$

gives $g(g(x)) = g(x)$, where $g(x) = f(f(x))$ for all $x \in R$. Of course g is also a differentiable function on R and $g(x) \ge 0$ for all $x \in R$. Then the range $T = g(R)$ of g is an interval contained in $[0, +\infty)$. Let a be the infimum of T. Since $g(t) = t$ for all $t \in T$ and g is continuous, it follows that $g(a) = a$. Assuming that T has more than one element, choose $\delta > 0$ such that $(a, a+\delta) \subseteq T$. Then $x \in (a-\delta, a)$ implies $g(x) \ge g(a)$ $(= a)$, hence

Therefore, $\frac{g(x)-g(a)}{x-a} \le 0$.

$$g'_L(a) = \lim_{x \to a^-} \frac{g(x)-g(a)}{x-a} \le 0. \quad \ldots(ii)$$

For $x \in (a, a+\delta)$ we have

$$\frac{g(x)-g(a)}{x-a} = 1.$$

Hence, $g'_R(a) = \lim_{x \to a^+} \frac{g(x)-g(a)}{x-a} = 1.$ $\quad \ldots(iii)$

(ii) and (iii) are contradictory since g is differentiable at a. We are led to the conclusion that T is a single

MAGAZINE KING

point, *i.e.*, g is a constant function, say $g(x) = c$ for all $x \in R$. This gives, using (i), that $f(c) = f(x)$ for all $x \in R$, showing that f is a constant function. Thus there is no nonconstant differentiable function satisfying (i).

5. (a) There are exactly 3 solutions, given by

$(p, q, r) = (2, 4, 13), (2, 5, 8)$ and $(3, 3, 7)$.

Note first that

$$(p-1)(q-1)(r-1) = pqr - (pq + qr + rp) + (p + q + r) - 1$$

which becomes, using the given equation,

$$(p-1)(q-1)(r-1) = 2(p + q + r - 1). \quad \ldots(i)$$

If $p \geq 4$, then $4 \leq p \leq q \leq r$ implies

$(p-1)(q-1)(r-1) \geq 9(r-1)$ and

$$2(p + q + r - 1) \leq 2(3r - 1);$$

and since, $9(r-1) - 2(3r-1) = 3r - 7 > 0$,

(i) cannot hold in this case. Thus $p < 4$. Since $p = 1$ clearly does not satisfy (i), we have $p = 2$ or 3. When $p = 2$, (i) becomes $(q-1)(r-1) = 2(q + r + 1)$ or $(q-3)(r-3) = 10$. Thus $q - 3 = 1$, $r - 3 = 10$ or $q - 3 = 2$, $r - 3 = 5$, yielding two solutions: $(2, 4, 13)$ and $(2, 5, 8)$. When $p = 3$, (i) becomes $(q-1)(r-1) = q + r + 2$ or $(q-2)(r-2) = 5$ which yields the third solution: $(3, 3, 7)$.

(b) The value is $\pi/4$ in all cases. To see this, set $A = \tan^{-1}(1/p)$, $B = \tan^{-1}(1/q)$ and $C = \tan^{-1}(1/r)$

Since $0 < \frac{1}{r} \leq \frac{1}{q} \leq \frac{1}{p} < 1$,

We have, $0 < C \leq B \leq A < \pi/4$

$\therefore \quad 0 < A + B + C < 3\pi/4.$

From the well known formula for $\tan(x + y)$ one easily deduces that for all x, y, z with $x + y + z \neq k\pi + \pi/2$ (where k denotes an integer),

$$\tan(x+y+z) = \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - (\tan x \tan y + \tan y \tan z + \tan z \tan x)}$$

$\therefore \quad \tan(A + B + C)$

$$= \frac{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - \frac{1}{pqr}}{1 - \left(\frac{1}{pq} + \frac{1}{qr} + \frac{1}{rp}\right)} = \frac{pq + qr + rp - 1}{pqr - (p+q+r)} = 1.$$

Hence, $A + B + C = \pi/4$

$$\therefore \quad \tan^{-1}(1/p) + \tan^{-1}(1/q) + \tan^{-1}(1/r) = \frac{\pi}{4}.$$

6. Since, $1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

we see that when $k = 0$,

$$(x_1, x_2, \ldots, x_n; y) = \left(1, 2, \ldots, n; \frac{n(n+1)}{2}\right)$$

is a solution. To see that we can generate infinitely many solutions. In general, set $c = (n(n+1))/2$ and notice that for all positive integers q, we have:

$$(c^k q^{3k+2})^3 + (2c^k q^{3k+2})^3 + \ldots + (nc^k q^{3k+2})^3$$

$$= c^{3k} q^{3(3k+2)} (1^3 + 2^3 + \ldots + n^3)$$

$$= c^{3k} q^{3(3k+2)} \left(\frac{n(n+1)}{2}\right)^2$$

$$= c^{3k+2} q^{3(3k+2)} = (cq^3)^{3k+2}.$$

That is, $(x_1, x_2, \ldots, x_n; y) = (c^k q^{3k+2}, 2c^k q^{3k+2}, \ldots, nc^k q^{3k+2}; cq^3)$ is a solution. This completes the proof.

7. Note first that $u_1 = 1 - u$. Since for all $x \in [u, 1]$, $u \leq x$ and $1 - x \leq 1 - u$ we have

$$1 - (\sqrt{ux} + \sqrt{(1-u)(1-x)})^2$$

$$= 1 - ux - (1-u)(1-x) - 2\sqrt{ux(1-u)(1-x)}$$

$$= u + x - 2ux - 2\sqrt{ux(1-u)(1-x)}$$

$$\leq u + x - 2ux - 2u(1 - x) = x - u.$$

Therefore, $f(x) = 0$ if $0 \leq x \leq u$ ...(i)

and

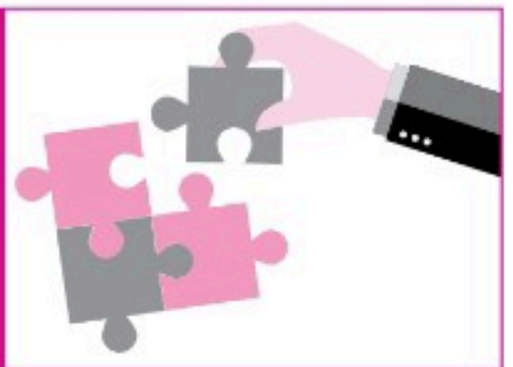
$f(x) \leq x - u$ if $u \leq x \leq 1$. ...(ii)

From (ii) we get $u_2 = f(u_1) - u = 1 - 2u$ if $u_1 \geq u$. An easy induction then yields $u_{n+1} = f(u_n) \leq u_n - u \leq 1 - (n+1)u$ if $u_i \geq u$ for all $i = 1, 2, \ldots, n$.

Thus for sufficiently large k, we must have $u_{k-1} < u$ and then $u_k = f(u_{k-1})$ 0 by (ii).

PUZZLE CORNER

ANSWER · NOVEMBER 2020



13+ 4	4÷ 1	10× 2	5	6 6	2÷ 3
3	4	5× 5	1	8+ 2	6
6	30× 3	4 4	8+ 2	5	1
2	5	1 1	6	12× 3	4
4− 5	15+ 6	3	3− 4	1	2 2
1	2 2	6	60× 3	4	5

YOU ASK WE ANSWER

Do you have a question that you just can't get answered?

Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough. The best questions and their solutions will be printed in this column each month.

1. There are n intermediate stations on a railway line from one terminus to another. In how many ways can the train stop at 3 of these intermediate stations if

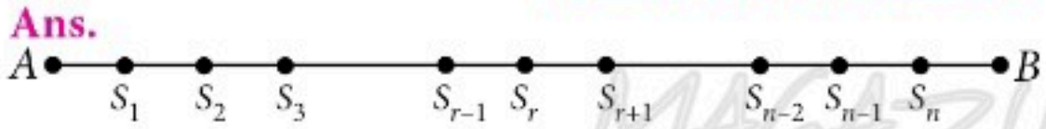
(i) all the three stations are consecutive?

(ii) at least two of the stations are consecutive?

(iii) no two of these stations are consecutive?

(Puneet Jain, Ahmedabad)

Ans.



(i) The number of triplets of consecutive stations, viz. $S_1S_2S_3, S_2S_3S_4, S_3S_4S_5, \ldots\ldots\ldots, S_{n-2}S_{n-1}S_n$ is $(n-2)$.

(ii) The total number of consecutive pair of stations, viz.

$S_1S_2, S_2S_3, \ldots\ldots\ldots, S_{n-1}S_n$ is $(n-1)$.

Each of the above pair can be associated with a third station in $(n-2)$ ways. Thus, choosing a pair of stations and any third station can be done in $(n-1)(n-2)$ ways. The above count also includes the case of three consecutive stations. However, we can see that each such case has been counted twice. For example, the pair S_4S_5 combined with S_6 and the pair S_5S_6 combined with S_4 are identical.

Hence, subtracting the excess counting, the number of ways in which three stations can be chosen so that at least two of them are consecutive is

$$(n-1)(n-2) - (n-2) = (n-2)^2.$$

(iii) Without restriction, the train can stop at any three stations in ${}^nC_3$ ways. Hence, the number of ways the train can stop so that no two stations are consecutive is

$${}^nC_3 - (n-2)^2 = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} - (n-2)^2$$

$$= (n-2)\left(\frac{n^2-n-6n+12}{6}\right) = \frac{(n-2)(n-3)(n-4)}{6} = {}^{n-2}C_3.$$

2. If a, b, c are in G.P. and the equation $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then show that $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P. *(Vishal, Dehradun)*

Ans. We have, $b^2 = ac$ ...(i)

The equation $ax^2 + 2bx + c = 0$

gives $x = \dfrac{-2b \pm \sqrt{4b^2 - 4ac}}{2a} = \dfrac{-b}{a}$ [using (i)]

and $dx^2 + 2ex + f = 0 \Rightarrow x = \dfrac{-e \pm \sqrt{e^2 - df}}{d}$

According to the given condition, we have

$$\frac{-b}{a} = \frac{-e \pm \sqrt{e^2 - df}}{d} \quad i.e. \quad \left(\frac{e}{d} - \frac{b}{a}\right)^2 = \frac{e^2 - df}{d^2}$$

$$i.e. \quad \frac{2be}{ad} = \frac{b^2}{a^2} + \frac{f}{d} = \frac{c}{a} + \frac{f}{d} \quad \text{[using (i)]}$$

$$i.e. \quad \frac{2be}{ac} = \frac{d}{a} + \frac{f}{c} \quad \left[\text{multiplying both sides by } \frac{d}{c}\right]$$

$$i.e. \quad \frac{2e}{b} = \frac{d}{a} + \frac{f}{c} \Rightarrow \frac{d}{a}, \frac{e}{b}, \frac{f}{c} \text{ are in A.P.}$$

3. Find the values of a, b, c if the following limit exists :

$$\lim_{x\to 0} \frac{ae^x - b\ln(1+x) + cxe^{-x}}{x^2 \sin x} = 2 \quad \textit{(Umang, Pune)}$$

Ans. Let $L = \lim\limits_{x\to 0} \dfrac{ae^x - b\ln(1+x) + cxe^{-x}}{x^2 \sin x}$

$$= \lim_{x\to 0} \frac{ae^x + cxe^{-x} - b\ln(1+x)}{x^3} \cdot \frac{x}{\sin x}$$

$$= \lim_{x\to 0} \frac{ae^x + cxe^{-x} - b\ln(1+x)}{x^3}$$

$$= \lim_{x\to 0} \frac{a\left(1 + x + \frac{x^2}{2!} + \ldots\right) + cx\left(1 - x + \frac{x^2}{2!} - \ldots\right) - b\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{a + (a + c - b)x + (a - 2c + b)\frac{x^2}{2} + \left(\frac{a}{3!} + \frac{c}{2!} - \frac{b}{3}\right)x^3 + \ldots}{x^3}$$

For a finite non-zero limit to exist, we have

$a = 0$ ...(i) $\quad a + c - b = 0$ ...(ii)

$a - 2c + b = 0$ ...(iii)

and $\frac{a}{6} + \frac{c}{2} - \frac{b}{3} = 2$ [according to the given condition]

i.e. $a + 3c - 2b = 12$ ...(iv)

Solving (i), (ii), (iii) and (iv), we can see that there is no value of a, b, c which satisfies all the equations.